

The arc space of the Grassmannian

Roi Docampo and Antonio Nigro

ABSTRACT. We study the arc space of the Grassmannian from the point of view of the singularities of Schubert varieties. Our main tool is a decomposition of the arc space of the Grassmannian that resembles the Schubert cell decomposition of the Grassmannian itself. Just as the combinatorics of Schubert cells is controlled by partitions, the combinatorics in the arc space is controlled by plane partitions (sometimes also called 3d partitions). A combination of a geometric analysis of the pieces in the decomposition and a combinatorial analysis of plane partitions leads to invariants of the singularities. As an application we reduce the computation of log canonical thresholds of pairs involving Schubert varieties to an easy linear programming problem. We also study the Nash problem for Schubert varieties, showing that the Nash map is always bijective in this case.

Introduction

Given a variety X , an arc on X is a germ of a parametrized curve, and the arc space $J_\infty X$ is a natural geometric object parametrizing arcs. Arc spaces have been featured repeatedly in recent years in algebraic geometry, from several points of view. They appear in singularity theory, mainly via the study of Nash-type problems [Nas95], a tool to understand resolutions of singularities. They are a key ingredient in the theory of motivic integration, introduced by Kontsevich [Kon95, DL99] and with many applications to the study of invariants of varieties and of singularities. And they are used in birational geometry in the study of singularities of pairs [Mus02, EM09].

The purpose of this paper is the study of the arc space of the Grassmannian and of its Schubert varieties.

Recall that the Grassmannian $G(k, n)$ is the space parametrizing k -dimensional vector subspaces in \mathbb{C}^n . This is a fundamental object in geometry, a source of many examples, and used often as the starting point in the construction of other varieties and invariants. The Schubert varieties appear as natural sub-objects inside the Grassmannian, and they provide a rich collection of singular varieties. These singularities have been studied thoroughly in the literature, in many different contexts.

The main tool that we propose is a decomposition of the arc space of the Grassmannian into pieces that we call *contact strata*. This stratification can be defined in two equivalent ways: either using orders of contact of arcs with respect to Schubert varieties, or using invariant factors of lattices naturally associated to arcs. The resulting pieces, the contact strata, should be thought as the arc space analogue of the Schubert cells of the Grassmannian itself. All subsets in the arc space which are relevant for the study of Schubert varieties (for example contact loci of Schubert varieties) can be

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decomposed using contact strata. This essentially reduces the computation of invariants to the understanding of contact strata.

For the study of contact strata we were inspired by previous work on the totally positive Grassmannian (mainly [FZ00]). In particular, *weighted planar networks* appear repeatedly in the paper. We use them to construct arcs, and to control (in a combinatorial way) the orders of contact of an arc with respect to Schubert varieties. Some special planar networks lead to a very explicit description of contact strata, which is useful throughout the paper. These combinatorial techniques allow us to classify contact strata, understand their basic geometry, and determine their position with respect to each other.

As a main application of our study, we give an effective algorithm to compute log canonical thresholds of pairs involving Schubert varieties. This is done by reducing the problem to maximizing a linear function on an explicit rational convex polytope, which we call the *polytope of normalized Schubert valuations*. After this is done, techniques from the theory of linear programming provide fast algorithms for the actual computation of the log canonical threshold.

For completeness, we also include the solution to the Nash problem for Schubert varieties in the Grassmannian. This turns out not to need a deep understanding of the arc space. We show that there exist resolutions (due to Zelevinsky) for which the exceptional components are in bijection with the irreducible components of the singular locus. This immediately implies that the Nash map is bijective, and that the Nash families are also in bijection with the components of the singular locus.

In the remainder of the introduction we give a more precise overview of the main results of the paper.

The stratification of the arc space. The \mathbb{C} -valued points of the arc space $J_\infty G(k, n)$ can be described using the defining universal property of the Grassmannian. They correspond to lattices $\Lambda \subset \mathbb{C}[[t]]^n$ for which the quotient $\mathbb{C}[[t]]^n / \Lambda$ is a free module of rank $n - k$. To get our stratification we classify these lattices according to their position with respect to a flag.

More precisely, start with a full flag in \mathbb{C}^n and consider the corresponding flag of lattices $0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = \mathbb{C}[[t]]^n$. For a given lattice $\Lambda \subset \mathbb{C}[[t]]^n$, the isomorphism type of the quotient module $\mathbb{C}[[t]]^n / (\Lambda + F_i)$ is determined by its invariant factors:

$$\frac{\mathbb{C}[[t]]^n}{\Lambda + F_i} \simeq \bigoplus_{j=1}^{n-k} \frac{\mathbb{C}[[t]]}{(t^{b_{i,j}})},$$

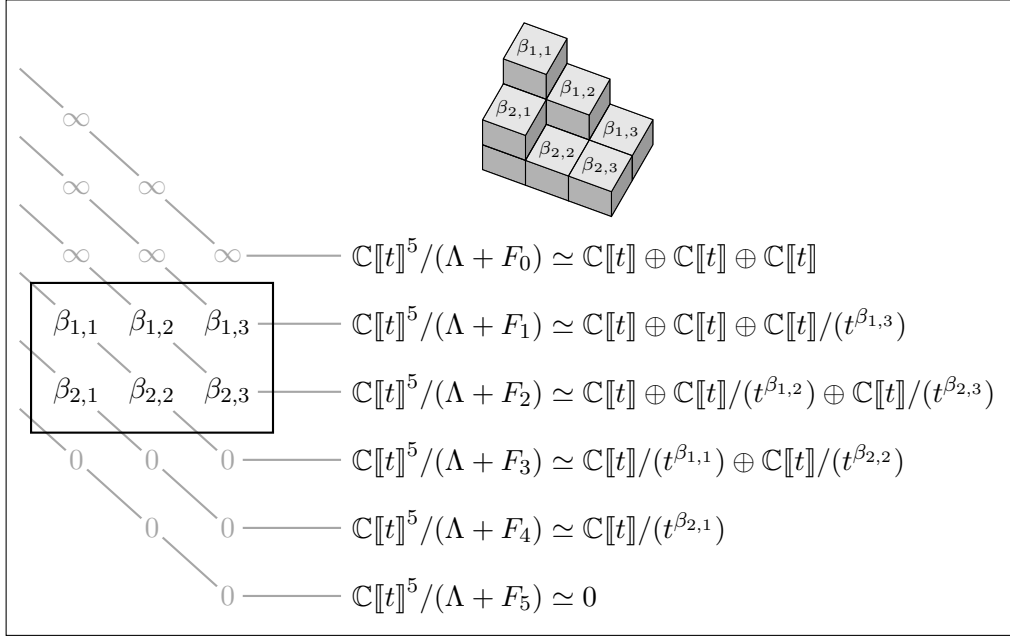
where the numbers $b_{i,j} \in \{0, 1, \dots, \infty\}$ verify $b_{i,j} \geq b_{i,j+1}$, and we use the convention $t^\infty = 0$. We consider the numbers $\beta_{i,j}$ given by

$$\beta_{i+k-n+j,j} = b_{i,j}.$$

In Theorem 4.1 we see that $\beta_{i,j} = \infty$ for $i \leq 0$ and $\beta_{i,j} = 0$ for $i > k$, so the only relevant values of $\beta_{i,j}$ occur for $1 \leq i \leq k$ and $1 \leq j \leq n - k$. In this way we obtain a matrix $\beta = (\beta_{i,j})$ of size $k \times (n - k)$, which we call the *invariant factor profile* of the arc Λ . See Fig. 0A for a diagram explaining the meaning of the numbers $\beta_{i,j}$ in the particular case of $G(2, 5)$.

We have a decomposition

$$J_\infty G(k, n) = \bigcup_{\beta} \mathcal{C}_\beta,$$


 FIGURE 0A. The meaning of the invariant factor profile for $G(2, 5)$.

where \mathcal{C}_β is the collection of arcs with invariant factor profile β . The pieces \mathcal{C}_β are called *contact strata*. The main facts about invariant factor profiles and contact strata are summarized in the following theorem.

Theorem 0.1.

- (1) *The invariant factor profile of an arc is determined by the orders of contact of the arc with respect to the Schubert varieties, and vice versa (Theorem 4.1).*
- (2) *The invariant factor profile is a plane partition¹, i.e., $\beta_{i,j} \geq \beta_{i+1,j}$ and $\beta_{i,j} \geq \beta_{i,j+1}$ (Proposition 3.4 and Theorem 3.8).*
- (3) *Every plane partition is the invariant factor profile of some arc (Theorems 3.8 and 6.3).*
- (4) *Contact strata are irreducible (Theorem 7.1).*

Because of fact (1), the decomposition into contact strata is very relevant for the study of the singularities of Schubert varieties. For example, for the computation of log canonical thresholds (Theorem 0.5) we use that contact loci of Schubert varieties are unions of contact strata.

Among the above facts, the most delicate is (3). For its proof, we need to produce arcs with prescribed invariant factor profile, and we do not know of a simple way of achieving this. Section 6 is devoted to this issue. Here is where we start using weighted planar networks (as mentioned above, inspired by [FZ00]). With this construction, we are able to use combinatorial techniques to control the orders of contact with respect to Schubert varieties. The resulting description of contact strata is very explicit. For example, to prove fact (4) we use planar networks to describe the generic point of each contact stratum.

There is another natural stratification of $J_\infty G(k, n)$, considering orbits of the action of the group of arcs $J_\infty B$, where $B \subset \text{GL}_n$ is the Borel subgroup. An analysis of this orbit

¹In the literature, plane partitions are sometimes called 3d partitions.

decomposition would be in the lines of previous approaches to the study of arc spaces. For example, this is the main idea used in the cases of toric varieties [IK03, Ish04] and of determinantal varieties [Doc13]. But in our case we found that the structure of the orbits is too complex for our study (see Section 5). Contact strata provide a coarser decomposition, simpler to understand, and enough for our purposes.

Geometry of contact strata. In order to compute invariants it is not enough to just stratify the arc space, we need to understand the geometry of each of the strata, and to study how these pieces are placed with respect to each other. From our point of view, we consider the following to be the main question.

Problem 0.2 (Nash problem for contact strata). *Given two plane partitions β and β' , determine whether there is a containment $\overline{\mathcal{C}}_\beta \supset \overline{\mathcal{C}}_{\beta'}$.*

Section 8 is devoted to the study of Problem 0.2. We are able to give an answer in several cases by analyzing the combinatorics of plane partitions. The main results are Lemma 8.3, which gives a necessary condition for a containment to exist, Theorems 8.5 and 8.7, which give sufficient conditions, and Proposition 8.4, which gives a complete answer for $G(2, 4)$. Again, for these results we often use planar networks to transform geometric questions into combinatorics. A general answer to Problem 0.2 seems very difficult.

Despite the fact that our answer to Problem 0.2 is only partial, we are able to use it effectively to compute invariants of contact strata. Namely, we prove the following result.

Theorem 0.3. *The codimension of a contact stratum \mathcal{C}_β in $J_\infty G(k, n)$ is the number of boxes in the plane partition β :*

$$\text{codim}(\mathcal{C}_\beta, J_\infty G(k, n)) = \sum_{i,j} \beta_{i,j}$$

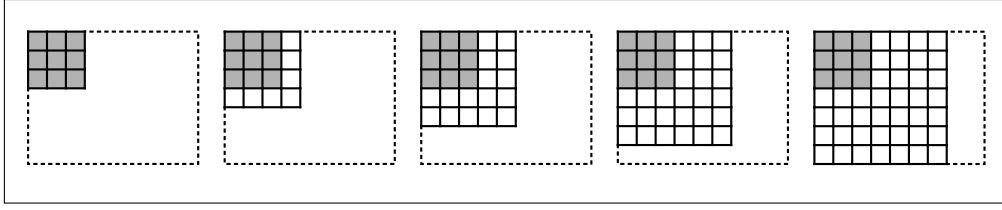
This theorem is proven studying chains of containments of closures of contact strata, and these are provided by our answers to Problem 0.2. The codimensions of contact strata immediately give log discrepancies of the corresponding valuations (which we call Schubert valuations, see Section 7), and the computation of log canonical thresholds gets reduced to the analysis of the combinatorics of plane partitions.

Log canonical thresholds of Schubert varieties. The Schubert varieties inside $G(k, n)$ are indexed by partitions $\lambda = (\lambda_1 \lambda_2 \dots)$ with at most k parts of size at most $n - k$. We denote them Ω_λ . A partition of the form $\lambda = (b^a) = (b.^a.b)$ is called rectangular. The following result is proven in Section 10.

Theorem 0.4. *Let Ω_λ be a Schubert variety in $G(k, n)$, and assume that $\lambda = (b^a)$ is rectangular. Consider $\lambda^s = ((b+s)^{a+s})$, and let $|\lambda^s| = (a+s)(b+s)$ denote the number of boxes in λ^s . Let $r = \min\{k-a, n-k-b\}$. Then the log canonical threshold of the pair $(G(k, n), \Omega_\lambda)$ is*

$$\text{lct}(G(k, n), \Omega_\lambda) = \min_{s=0 \dots r} \left\{ \frac{|\lambda^s|}{s+1} \right\} = \min_{s=0 \dots r} \left\{ \frac{(a+s)(b+s)}{s+1} \right\}.$$

It should be noted that Schubert varieties corresponding to rectangular λ are essentially generic determinantal varieties, and therefore their log canonical thresholds were already known, see [Joh03, Doc13]. But the proof that we provide is new, and also gives a natural combinatorial interpretation for the numbers appearing in the formula.


 FIGURE 0B. The partitions λ^s in Theorem 0.4 for $\lambda = (333)$ in $G(7, 16)$.

The partitions λ^s in Theorem 0.4 are obtained from λ by adding rims of boxes, without exceeding the maximal allowed size (the rectangle with k rows and $(n - k)$ columns). For example, the case of $\lambda = (333)$ in $G(7, 16)$ appears in Fig. 0B. For the log canonical threshold we get:

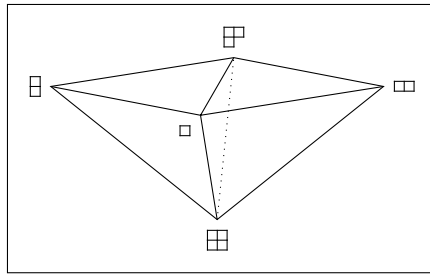
$$\text{lct}(G(7, 16), \Omega_{(333)}) = \min \left\{ \frac{9}{1}, \frac{16}{2}, \frac{25}{3}, \frac{36}{4}, \frac{49}{5} \right\} = \frac{16}{2} = 8.$$

For more general Schubert varieties (when λ is not necessarily rectangular), we have an analogue version of Theorem 0.4 which expresses the Arnold multiplicity (the reciprocal of the log canonical threshold) as the maximum of a linear function on (the extremal points of) a rational convex polytope.

Let $\mathbb{RPP}(k, n)$ the convex hull of the set of plane partitions inside $\mathbb{R}^{k(n-k)}$. Then $\mathbb{RPP}(k, n)$ is a pointed rational convex polyhedral cone with vertex at the origin. For a point $\beta \in \mathbb{RPP}(k, n)$, we denote by $|\beta| = \sum \beta_{i,j}$ the *volume* of β , and we let $\text{SV}(k, n)$ be the subset of $\mathbb{RPP}(k, n)$ containing elements of volume 1:

$$\text{SV}(k, n) = \{ \beta \in \mathbb{RPP}(k, n) \mid |\beta| = 1 \}.$$

We call $\text{SV}(k, n)$ the *polytope of normalized Schubert valuations*. The structure of $\text{SV}(k, n)$ is very explicit. It is a bounded rational convex polytope whose vertices are in bijection with non-empty partitions with at most k parts of size at most $n - k$. It has a natural simplicial structure, where the r -dimensional simplices correspond to chains of partitions $\lambda^0 \subsetneq \lambda^1 \subsetneq \dots \subsetneq \lambda^r$. See Fig. 0C for the example of $G(2, 4)$.


 FIGURE 0C. The simplicial structure on $\text{SV}(2, 4)$.

Given a partition λ , the order of contact with respect to Ω_λ induces a function on $\text{SV}(k, n)$:

$$\text{ord}(\lambda): \text{SV}(k, n) \rightarrow \mathbb{R}, \quad \beta \mapsto \text{ord}(\lambda)(\beta) = \text{ord}_\beta(\Omega_\lambda).$$

The function $\text{ord}(\lambda)$ can be described explicitly, by considering the corners of the partition and the half diagonals emanating from these corners. We refer to Section 10 for details; see Fig. 0D for some examples. From this description it follows that $\text{ord}(\lambda)$ is

a concave piecewise-linear function. We denote by $H_\lambda \subset \mathbb{R}^{k(n-k)}$ the linear subspace obtained as the zero locus of the linear equations defining $\text{ord}(\lambda)$. Notice that H_λ is the biggest linear space contained in the corner locus of $\text{ord}(\lambda)$, and in particular $\text{ord}(\lambda)$ is linear on H_λ .

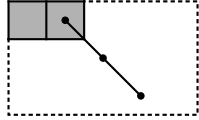
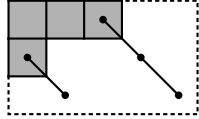
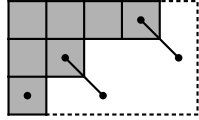
Partition	Diagonals	$\text{ord}(\lambda)$	Equations of H_λ
(2)		$\beta_{1,2} + \beta_{2,3} + \beta_{3,4}$	None
(31)		$\min \begin{Bmatrix} \beta_{1,3} + \beta_{2,4} + \beta_{3,5} \\ \beta_{2,1} + \beta_{3,2} \end{Bmatrix}$	$\beta_{1,3} + \beta_{2,4} + \beta_{3,5} = \beta_{2,1} + \beta_{3,2}$
(421)		$\min \begin{Bmatrix} \beta_{1,4} + \beta_{2,5} \\ \beta_{2,2} + \beta_{3,3} \\ \beta_{3,1} \end{Bmatrix}$	$\beta_{1,4} + \beta_{2,5} = \beta_{2,2} + \beta_{3,3} = \beta_{3,1}$

FIGURE 0D. Examples of $\text{ord}(\lambda)$ and H_λ in $G(3, 8)$.

Theorem 0.5. *Let Ω_λ be a Schubert variety in $G(k, n)$. Then the Arnold multiplicity of the pair $(G(k, n), \Omega_\lambda)$ is the maximum of $\text{ord}(\lambda)$ on $\text{SV}(k, n) \cap H_\lambda$.*

Notice that $\text{SV}(k, n) \cap H_\lambda$ is a rational convex polytope, and in particular the maximum of $\text{ord}(\lambda)$, which is linear on the polytope, is achieved on an extremal point. For example, when $\lambda = (b^a)$ is rectangular, the partitions $\lambda^0, \dots, \lambda^r$ appearing in Theorem 0.4 give some of the extremal points of $\text{SV}(k, n) \cap H_\lambda$ (in the rectangular case, the other extremal points are easy to discard).

From Theorem 0.5, to obtain an actual value for the Arnold multiplicity (and hence for the log canonical threshold) one would normally use a computer. The problem of maximizing a linear function on a convex polytope is the subject of *linear programming*. This is a highly developed theory, providing several very efficient algorithms to calculate both approximate and exact solutions. Both the polytope $\text{SV}(k, n) \cap H_\lambda$ and the linear function $\text{ord}(\lambda)$ are straightforward to describe to a computer, and in practice we found that the standard libraries dedicated to linear programming are very fast at computing log canonical thresholds, even for large values of k and n and complicated partitions λ .

1. Generalities on arc spaces

In this section we review the theory of arc spaces. For a full treatment, including proofs, we refer the reader to [ELM04, Voj07, Ish08, dFEI08, Mor09, EM09].

Basic conventions. We work over the complex numbers \mathbb{C} , although most of our results would be valid after replacing \mathbb{C} with an arbitrary algebraically closed field. All schemes are quasi-compact, quasi-separated, and defined over \mathbb{C} , but not necessarily Noetherian. By variety we mean a separated, reduced, and irreducible scheme of finite type over \mathbb{C} . All morphisms of schemes are defined over \mathbb{C} .

Arcs and jets. Fix a scheme X . An *arc* γ on X is a morphism

$$\gamma: \operatorname{Spec} \mathbb{C}[[t]] \rightarrow X.$$

Similarly, for a non-negative integer m , a *jet* γ on X of order m is a morphism

$$\gamma: \operatorname{Spec} \mathbb{C}[t]/(t^{m+1}) \rightarrow X.$$

Notice that a jet of order 1 is a tangent vector. More generally, for a \mathbb{C} -algebra A , morphisms of the type

$$\operatorname{Spec} A[[t]] \rightarrow X \quad \text{and} \quad \operatorname{Spec} A[t]/(t^{m+1}) \rightarrow X$$

are called A -valued arcs and jets on X . An A -valued arc/jet should be thought as a family of arcs/jets parametrized by $\operatorname{Spec} A$.

We denote by 0 the closed points of $\operatorname{Spec} \mathbb{C}[[t]]$ and $\operatorname{Spec} \mathbb{C}[t]/(t^{m+1})$, and by η the generic point of $\operatorname{Spec} \mathbb{C}[[t]]$. For an arc or jet γ , the point $\gamma(0)$ is called the *center*, *origin*, or *special point* of γ . If γ is an arc, $\gamma(\eta)$ is called the *generic point* of γ . This terminology is also used for K -valued arcs and jets, where K is a field extension of \mathbb{C} . A K -valued γ arc is called *fat* if its generic point $\gamma(\eta)$ is the generic point of X ; otherwise it is called *thin*.

Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a sheaf of ideals in X . For an arc or jet γ , the inverse image $\gamma^{-1}(\mathfrak{a})$ is an ideal of the form (t^e) , for some number $e \in \{0, 1, \dots, \infty\}$. Here we use the convention $t^\infty = 0$ to cover the case where the inverse image is the zero ideal. This number e is called the *order of contact* of γ along the ideal \mathfrak{a} , and denoted $\operatorname{ord}_\gamma(\mathfrak{a})$. If Y is the closed subscheme of X defined by \mathfrak{a} , we also write $\operatorname{ord}_\gamma(Y)$ for this order.

Arc spaces and jet schemes. The *arc space* of X is the universal object parametrizing families of arcs on X . It is denoted $J_\infty X$ and it is characterized² by its functor of points:

$$J_\infty X(A) = \operatorname{Hom}_{\mathbb{C}\text{-Schemes}}(\operatorname{Spec} A[[t]], X).$$

Similarly, the *jet scheme* of order m of X , denoted $J_m X$ is given by

$$J_m X(A) = \operatorname{Hom}_{\mathbb{C}\text{-Schemes}}(\operatorname{Spec} A[t]/(t^{m+1}), X).$$

It can be shown that the arc space and the jet schemes are schemes. If X is of finite type, the jet schemes are also of finite type, but the arc space is not (unless X is zero-dimensional).

The natural quotient maps at the level of algebras

$$A[[t]] \rightarrow A[t]/(t^{m+1}) \rightarrow A$$

induce morphisms of schemes:

$$J_\infty X \rightarrow J_m X \rightarrow X$$

These morphisms are affine, and are called the *truncation maps*. The arc space is the projective limit of the jet schemes via the truncation maps.

There are natural sections of the truncation maps at level zero:

$$X \rightarrow J_\infty X \quad \text{and} \quad X \rightarrow J_m X.$$

The images of these sections are called the *constant arcs* and jets. In general there are no natural sections of the truncations $J_\infty X \rightarrow J_m X$ for $m \geq 1$.

²Usually the arc space is defined as the projective limit of the jet schemes, so its functor of points is in principle different from the one written here. But it follows from [Bha, Corollary 1.2] that both definitions agree.

The construction of arc spaces and jet schemes is functorial. Given a morphism of schemes $f: X \rightarrow Y$, composition with f induces natural morphisms at the level of arc spaces and jet schemes. These morphisms are compatible with the truncation maps:

$$\begin{array}{ccc} J_\infty X & \longrightarrow & J_\infty Y \\ \downarrow & & \downarrow \\ J_m X & \longrightarrow & J_m Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

As a consequence of functoriality, if G is a group scheme, the arc space $J_\infty G$ and jet schemes $J_m G$ are also groups. Moreover, if G acts on X , we get induced actions of $J_\infty G$ on $J_\infty X$, and of $J_m G$ on $J_m X$. All these groups structures and actions are compatible with the truncation maps.

Constructible sets and contact loci. From now on we assume that X is a variety. For the general definition of constructible set in a scheme, we refer the reader to [Gro61, 0_{III}, §9.1]. In the finite type case (for the variety X and for the jet schemes $J_m X$) this is the familiar notion: a set is constructible if it is a finite boolean combination of Zariski closed subsets. For the arc space a constructible set turns out to be the same as a *cylinder* [ELM04]: the inverse image via a truncation map of a constructible set in some jet scheme $J_m X$.

The most important examples of constructible sets in the arc space are contact loci. Given a closed subscheme $Y \subset X$ and a number $p \in \{0, 1, \dots\}$, we define

$$\text{Cont}^{\geq p}(Y) = \{ \gamma \in J_\infty X \mid \text{ord}_\gamma(Y) \geq p \}.$$

We also define $\text{Cont}^{=p}(Y)$ in the obvious way, and analogous versions in the jet schemes: $\text{Cont}_m^{\geq p}(Y)$ and $\text{Cont}_m^{=p}(Y)$. We call these types of sets *contact loci*. Notice that a contact locus in the arc space is the inverse image of a contact locus in a jet scheme of high enough order (the order of an arc is determined by the order of a high enough truncation). In particular, contact loci are constructible.

Valuations. Let R be an integral domain containing \mathbb{C} . A (discrete, rank at most one) *semi-valuation* on R is a function $v: R \rightarrow \{0, 1, \dots, \infty\}$ satisfying the following properties:

- (1) $v(fg) = v(f) + v(g)$ for all elements $f, g \in R$,
- (2) $v(f + g) \geq \min\{v(f), v(g)\}$ for all elements $f, g \in R$,
- (3) $v(z) = 0$ for all non-zero constants $z \in \mathbb{C} \setminus \{0\}$, and
- (4) $v(0) = \infty$.

We say that v is a *valuation* if furthermore:

- (5) $v(f) = \infty$ if and only if $f = 0$.

We say that v is *trivial* if its only values are 0 and ∞ . The greatest common divisor of the non-zero values of a non-trivial semi-valuation v is called its *multiplicity*, and denoted q_v . The prime ideals

$$\mathfrak{h}_v = \{f \in R \mid v(f) = \infty\} \quad \text{and} \quad \mathfrak{c}_v = \{f \in R \mid v(f) > 0\}$$

are called the *home* and *center* of v . A semi-valuation is a valuation precisely when its home is zero. A semi-valuation v induces a valuation in the standard sense in the field of

fractions $\text{Frac}(R/\mathfrak{b}_v)$. The corresponding *valuation ring* is denoted by $\mathcal{O}_v \subset \text{Frac}(R/\mathfrak{b}_v)$. Notice that \mathcal{O}_v is either a field (if v is trivial) or a discrete valuation ring of rank one.

Let R_f be the localization of R obtained by inverting f . Then the set of semi-valuations of R_f is in natural bijection with the set of semi-valuations of R for which f has value zero. This allows us to glue this construction, and talk about *semi-valuations* and *valuations* on a variety X . Geometrically, a semi-valuation on X can be thought as a choice of a subvariety $Y \subset X$ (the home of the semi-valuation) and a valuation (in the standard sense) on Y .

Semi-valuations and arcs are closely related. Let γ be a point of $J_\infty X$ in the sense of schemes, and let K_γ be its residue field. It corresponds to a K_γ -valued arc on X :

$$\gamma: \text{Spec } K_\gamma[[t]] \rightarrow X.$$

Then ord_γ is a semi-valuation on X . Its home is $\gamma(\eta)$, the generic point of the arc. Its center in the sense of semi-valuations agrees with $\gamma(0)$, the center of γ in the sense of arcs. It is trivial if and only if γ is a constant arc, and it is a valuation if and only if γ is fat. More geometrically, this construction can be reduced to use only \mathbb{C} -valued arcs. We consider the closure of γ in the arc space, denoted $\mathcal{C} = \overline{\{\gamma\}} \subset J_\infty X$. Then the semi-valuation ord_γ can be recovered from the semi-valuations of the arcs in the family \mathcal{C} :

$$\text{ord}_\gamma = \text{ord}_\mathcal{C} = \min\{\text{ord}_\alpha \mid \alpha \in \mathcal{C}\}.$$

Conversely, every semi-valuation is induced by some arc. Let v be a non-trivial semi-valuation on X with multiplicity q_v , and consider its valuation ring \mathcal{O}_v . The completion $\widehat{\mathcal{O}}_v$ is isomorphic to the power series ring $\widehat{K}_v[[t]]$, where \widehat{K}_v is the residue field of $\widehat{\mathcal{O}}_v$. For a choice φ of any such isomorphism we get a \widehat{K}_v -valued arc $\gamma_{v,\varphi}$:

$$\begin{array}{ccccccc} \text{Spec } \widehat{K}_v[[t]] & \xrightarrow{t \mapsto t^{q_v}} & \text{Spec } \widehat{K}_v[[t]] & \xrightarrow{\varphi} & \text{Spec } \widehat{\mathcal{O}}_v & \longrightarrow & \text{Spec } \mathcal{O}_v \longrightarrow X. \\ & & & & & \searrow \gamma_{v,\varphi} & \nearrow \end{array}$$

It is straightforward to check that $\text{ord}_{\gamma_{v,\varphi}} = v$. Trivial valuations can be written as $v = \text{ord}_\gamma$, where γ is any constant arc for which $\gamma(0)$ is the home of v .

Among all arcs giving the same semi-valuation v , there is a distinguished one, characterized by being maximal with respect to specialization in the arc space. Namely, we consider the family

$$\mathcal{C}_v = \overline{\{\gamma \in J_\infty X \mid \text{ord}_\gamma = v\}}.$$

Then \mathcal{C}_v is irreducible [ELM04, Ish08, dFEI08, Mor09], and its generic point γ_v verifies $\text{ord}_{\gamma_v} = v$. Any other arc inducing v is a specialization of γ_v . Following the terminology of [Mor09], we call \mathcal{C}_v the *maximal arc set* associated to v . If v is trivial, we have that $\mathcal{C}_v = J_\infty Y$, where Y is the home of v .

Divisorial valuations. Among all valuations on a variety X , the *divisorial* ones are of particular importance. Let $f: Y \rightarrow X$ be a proper birational map with Y smooth, and let E be a prime divisor on Y . Then computing orders of vanishing along E gives a valuation on X , which we denote val_E . Any valuation of the form $q \cdot \text{val}_E$, where q is a positive integer, is called a *divisorial valuation* on X . The maximal arc sets associated to divisorial valuations are called *maximal divisorial sets*.

One of the main results of [ELM04] and [dFEI08] is a characterization of divisorial valuations among all semi-valuations using contact loci. In precise terms, they prove that the following are equivalent:

- (1) v is a divisorial valuation;

- (2) there exists a contact locus \mathcal{C} such that $v = \text{ord}_{\mathcal{C}}$; and
- (3) there exists a constructible set \mathcal{C} such that $v = \text{ord}_{\mathcal{C}}$.

Moreover, for a subset $\mathcal{C} \subset J_{\infty}X$, the following are also equivalent:

- (1) \mathcal{C} is a maximal divisorial set; and
- (2) \mathcal{C} is a fat irreducible component of a contact locus.

For a divisorial valuation $v = q \cdot \text{val}_E$, the corresponding maximal divisorial set \mathcal{C}_v has an explicit geometric interpretation. It is the closure of the set of arcs whose lift to Y is tangent to E with order $\geq q$. In symbols:

$$\mathcal{C}_v = \overline{\{f(\gamma) \mid \gamma \in J_{\infty}Y, \text{ord}_{\gamma}(E) \geq q\}}.$$

Discrepancies and log canonical thresholds. The importance of arc spaces from the point of view of the minimal model program resides in a formula that computes discrepancies of divisorial valuations in terms of arcs. We restrict ourselves to the smooth case, which will be enough for our purposes. Let X be a smooth variety, and consider a divisorial valuation $v = q_v \cdot \text{val}_E$, where E is a prime divisor in some smooth birational model $f: Y \rightarrow X$. Then the *discrepancies* of E and v are defined as

$$k_E(X) = \text{ord}_E(K_{Y/X}), \quad \text{and} \quad k_v(X) = q_v \cdot \text{ord}_E(K_{Y/X}),$$

where $K_{Y/X} \sim K_Y - f^*(K_X)$ is the relative canonical divisor. A standard computation shows that $k_v(X)$ does not depend on the choice of model Y . We have the following formula [Mus01, ELM04, dFEI08]:

$$q_v + k_v(X) = \text{codim}(\mathcal{C}_v, J_{\infty}X). \quad (1a)$$

Since we assume that X is smooth, the codimension in the above formula can be computed either in the sense of the Zariski topology of $J_{\infty}X$, or in the sense of cylinders (as the codimension of a high enough truncation).

Using discrepancies we can define the *log canonical threshold*, an invariant of the singularities of a pair which is central in the minimal model program. Let $Z \subset X$ be a subscheme, and consider a log resolution of the pair (X, Z) . This consists of a proper birational map $f: Y \rightarrow X$ where Y is smooth, the scheme theoretic inverse image of Z is a divisor A , and $A + \text{Ex}(f)$ is a divisor with simple normal crossings. Then the *log canonical threshold* of the pair (X, Z) is defined as

$$\text{lct}(X, Z) = \min_E \left\{ \frac{1 + k_E(X)}{\text{val}_E(Z)} \right\}.$$

In this formula E ranges among the prime exceptional divisors of f . As above, one can show that $\text{lct}(X, Z)$ does not depend on the choice of log resolution Y . Using arc spaces we can express the formula for the log canonical threshold in the following way:

$$\text{lct}(X, Z) = \min_{\mathcal{C}} \left\{ \frac{\text{codim}(\mathcal{C}, J_{\infty}X)}{\text{ord}_{\mathcal{C}}(Z)} \right\}. \quad (1b)$$

Here \mathcal{C} ranges in principle among all maximal divisorial sets of $J_{\infty}X$, but one can easily show that it is enough to consider only the irreducible components of all the contact loci $\text{Cont}^{\geq p}(Z)$.

For us it will sometimes be more convenient to deal with the *Arnold multiplicity*, which is just the reciprocal of the log canonical threshold:

$$\text{Arnold-mult}(X, Z) = \max_E \left\{ \frac{\text{val}_E(Z)}{1 + k_E(X)} \right\} = \max_{\mathcal{C}} \left\{ \frac{\text{ord}_{\mathcal{C}}(Z)}{\text{codim}(\mathcal{C}, J_{\infty}X)} \right\}. \quad (1c)$$

Nash-type problems. Let X be a variety, and denote by $\text{Sing}(X)$ its singular locus. The fat irreducible components of the contact locus

$$\text{Cont}^{\geq 1}(\text{Sing}(X)) \subset J_{\infty}X.$$

are called the *Nash families* of arcs of X . From the above discussion, we see that the Nash families are the maximal divisorial sets associated to some divisorial valuations, which we call the *Nash valuations* of X . The Nash valuations, apart from being divisorial, are also *essential*, in the sense that they appear as irreducible components of the exceptional locus of every resolution of singularities of X . This is what is known as the *Nash map*:

$$\{\text{Nash valuations of } X\} \subseteq \{\text{essential valuations of } X\}.$$

The *Nash problem*, in its more general form, asks for a geometric characterization of the image of the Nash map. The *Nash conjecture* asserts that the Nash map is a bijection.

The Nash problem has a long history. The Nash conjecture turns out to be true for curves, for surfaces [FdBPP12, dFD15], and for several special families of singularities in higher dimensions, including toric varieties [IK03, Ish05, Ish06, GP07, PPP08, LJR12, LA11, LA16]. But there are counterexamples to the Nash conjecture in all dimensions ≥ 3 [IK03, dF13, JK13]. For an approach to the Nash problem in higher dimensions using the minimal model program, see [dFD15].

We will also be interested in a variant of the Nash problem that we call the *generalized Nash problem*. The above construction of the maximal arc set associated to a semi-valuation can be thought as an inclusion:

$$\{\text{semi-valuations on } X\} \subseteq J_{\infty}X,$$

where a semi-valuation v gets sent to the generic arc in \mathcal{C}_v . This endows the set of semi-valuations with a geometric structure. As the Nash problem and the formula for discrepancies show, this structure is relevant from the point of view of singularity theory. A basic question in this context is the following: given two semi-valuations v_1 and v_2 , determine whether there is an inclusion $\mathcal{C}_{v_1} \supseteq \mathcal{C}_{v_2}$. This is what we call the *generalized Nash problem*.

We understand the generalized Nash problem for invariant valuations on toric varieties [Ish08] and on determinantal varieties [Doc13]. But beyond this very little is known, even for valuations on the plane [Ish08, FdBPPPP].

2. The Grassmannian and its Schubert varieties

In this section we discuss generalities about Grassmannians and Schubert varieties. Our main purpose is to fix notation and recall basic results that will be used in the rest of the paper. All results are well-known, and we mostly enumerate them without proof. For details we refer the reader to any of the standard texts in the subject, for example [ACGH85, Chapter II], [BV88], or [Ful97].

Grassmannians. Fix integers $0 < k < n$. The *Grassmannian* of k -planes in \mathbb{C}^n is denoted by $G(k, n)$. A point $V \in G(k, n)$ can be described as the row span of a full-rank matrix with k rows and n columns:

$$V = \text{row span} \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kn} \end{pmatrix}, \quad v_{ij} \in \mathbb{C}.$$

Such a matrix is determined by V only up to left multiplication by an element of GL_k . This way we obtain an identification of $G(k, n)$ with the GIT quotient $\mathrm{GL}_k \backslash \mathrm{Mat}_{k \times n}$.

The group GL_n has a natural right action on $G(k, n)$. This identifies the Grassmannian with the quotient $P_{k,n} \backslash \mathrm{GL}_n$, where $P_{k,n}$ is the parabolic subgroup of GL_n whose elements have zeros in the lower-left block of size $(k) \times (n - k)$.

Three subgroups of GL_n will be featured prominently in the rest of the paper. The first one is the torus $T = (\mathbb{C}^*)^n$, the subgroup of diagonal matrices. The other two are the Borel subgroup $B = B^+$ and the opposite Borel subgroup B^- , containing, respectively, the upper- and lower-triangular matrices. Also relevant is the Weyl group $W = S_n$, the symmetric group on n letters, naturally embedded in GL_n as the group of permutation matrices.

We denote by $\{e_1, \dots, e_n\}$ the standard basis for \mathbb{C}^n . The torus-fixed points of $G(k, n)$ are determined by the k -element subsets of $\{e_1, \dots, e_n\}$. More precisely, given a multi-index $I = [i_1 \dots i_k]$, where $1 \leq i_1 < \dots < i_k \leq n$, we can consider the following point in $G(k, n)$:

$$V_I = \langle e_{i_1}, \dots, e_{i_k} \rangle.$$

Then the V_I are all the torus-fixed points in $G(k, n)$.

Schubert varieties. The *Schubert cells* are the Borel orbits in $G(k, n)$. The *Schubert varieties* are the closures of the Schubert cells. Each Borel orbit contains exactly one torus-fixed point. For a multi-index $I = [i_1 \dots i_k]$, we denote by Ω_I° the Schubert cell containing V_I , and by Ω_I the closure of Ω_I° . The Schubert cell $\Omega_{[1 \dots k]}^\circ$ is called the *big cell*.

Schubert varieties can be described more explicitly as follows. We consider the flag

$$F_1 \subset F_2 \subset \dots \subset F_n = \mathbb{C}^n,$$

where F_i is spanned by the last i vectors in the standard basis of \mathbb{C}^n . Notice that the Borel subgroup B is the stabilizer of F_\bullet . For a multi-index $I = [i_1 \dots i_k]$, the Schubert variety associated to I is the subset of $G(k, n)$ given by

$$\Omega_I = \{V \in G(k, n) \mid \dim V \cap F_{n+1-i_s} \geq k+1-s, \ 1 \leq s \leq k\}.$$

Bruhat order. To a multi-index $I = [i_1 \dots i_k]$ we associate the partition $\lambda = (\lambda_1 \dots \lambda_k)$ given by

$$i_s = s + \lambda_{k+1-s}.$$

Notice that $n - k \geq \lambda_1 \geq \dots \geq \lambda_k \geq 0$. This association induces a bijection between multi-indexes of length k in the range $\{1, \dots, n\}$, and partitions with at most k parts of size at most $n - k$. It is helpful to visualize partitions via the associated Ferrers-Young diagrams; some examples in $G(3, 6)$ are given in Fig. 2A.

I	[456]	[356]	[236]	[146]	[245]	[124]	[123]
λ	(333)	(332)	(311)	(32)	(221)	(1)	\emptyset

FIGURE 2A. Some multi-indexes, partitions, and diagrams in $G(3, 6)$.

If λ is the partition associated to a multi-index I , we also use the notations $\Omega_\lambda = \Omega_I$ and $\Omega_\lambda^\circ = \Omega_I^\circ$. Given two partitions λ and μ , the containment of the Schubert varieties $\Omega_\lambda \subseteq \Omega_\mu$ corresponds to the reversed containment of the (Ferrers-Young diagrams of

the) partitions $\lambda \supseteq \mu$. In terms of multi-indexes, given $I = [i_1 \dots i_k]$ and $J = [j_1 \dots j_k]$, the containment $\Omega_I \subseteq \Omega_J$ corresponds to $i_s \geq j_s$; in this situation we write $I \geq J$. See Fig. 2B for an example: the first two diagrams show the poset of Schubert varieties in $G(2, 4)$ using multi-indexes and partitions.

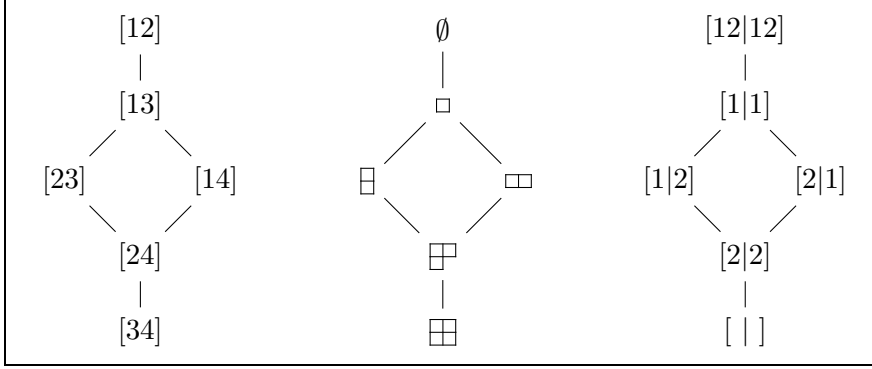


FIGURE 2B. The poset of Schubert varieties in $G(2, 4)$.

The codimension of Ω_λ in $G(k, n)$ is $|\lambda| = \lambda_1 + \dots + \lambda_k$, that is, the number of boxes in the diagram of λ . Moreover, Ω_λ° is isomorphic to $\mathbb{A}^{k(n-k)-|\lambda|}$.

Plücker coordinates. We consider a matrix

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{k1} & \cdots & X_{kn} \end{pmatrix}$$

where the entries are indeterminates. The $k \times k$ minors of this matrix are called the *Plücker coordinates* of $G(k, n)$. Given a tuple of indexes $I = [i_1 \dots i_k]$ (not necessarily distinct or in increasing order), the minor determined by the columns in I will also be denoted by $[i_1 \dots i_k]$. If $I = [i_1 \dots i_k]$ is a multi-index, then $[i_1 \dots i_k]$ is a Plücker coordinate. This abuse of notation (using the same symbols to denote a multi-index and a Plücker coordinate) will not cause problems.

Given a k -plane $V \subset \mathbb{C}^n$, we obtain a line $\wedge^k V \subset \wedge^k \mathbb{C}^n$. This induces the Plücker embedding $G(k, n) \hookrightarrow \mathbb{P}(\wedge^k \mathbb{C}^n)$. The homogeneous coordinate ring of $G(k, n)$ corresponding to the Plücker embedding will be denoted by $\mathbb{C}[G(k, n)]$; it is isomorphic to the subring of the polynomial ring $\mathbb{C}[X_{ij}]$ generated by the Plücker coordinates.

For a multi-index I with associated partition λ , we denote by $\mathcal{I}_I = \mathcal{I}_\lambda$ the ideal of Ω_I in $G(k, n)$. We think of \mathcal{I}_I as an ideal in $\mathbb{C}[G(k, n)]$. The following result is classic³.

Theorem 2.1. *Let $[i_1 \dots i_k]$ be a Plücker coordinate. Then the Plücker coordinates $[j_1 \dots j_k]$ such that $[j_1 \dots j_k] \not\geq [i_1 \dots i_k]$ generate the ideal $\mathcal{I}_{[i_1 \dots i_k]}$.*

In particular, Ω_\square is the divisor with equation $[1 \dots k]$, the determinant of the first k columns of X . The big cell Ω_\emptyset° is given by the non-vanishing of $[1 \dots k]$.

³The proof can be found in many places, for example in [BV88, Theorem 1.4]. But notice that the notation in [BV88] for Schubert varieties differs from ours. What they denote $\Omega(a_1, \dots, a_k)$ corresponds to our Ω_I , where $I = [i_1 \dots i_k]$ is given by $i_s = n + 1 - a_{k+1-s}$.

Plücker relations. For our analysis of the arc space of $G(k, n)$ we will need some understanding of the structure of products of ideals of Schubert varieties. In this study, the Plücker relations play an important role. For our purposes it will be enough to consider the following special case. For a proof, we refer the reader to [BV88, Lemma 4.4].

Theorem 2.2. *Consider tuples of indexes $[i_1 \dots i_k]$ and $[j_1 \dots j_k]$, and let u be an integer such that $1 \leq u \leq k$. Then*

$$[i_1 \dots i_k] \cdot [j_1 \dots j_k] = \sum_{v=1}^k \pm [i_1 \dots i_{u-1} j_v i_{u+1} \dots i_k] \cdot [j_1 \dots j_{v-1} i_u j_{v+1} \dots j_k].$$

The opposite big cell. Using the opposite Borel B^- , instead of B , we define *opposite Schubert cells* and *opposite Schubert varieties*. We denote them with an inverted circumflex, like $\check{\Omega}_I^\circ$ and $\check{\Omega}_\lambda$.

We are mainly interested in the *opposite big cell*, which we will denote by $\mathcal{U} = \check{\Omega}_{[n-k+1 \dots n]}^\circ$. It is given by the non-vanishing of the Plücker coordinate $[n-k+1 \dots n]$, the determinant of the last k columns of X . A point in \mathcal{U} is uniquely represented by a matrix of the form $(X_{\mathcal{U}} | \Delta')$, where:

$$X_{\mathcal{U}} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1(n-k)} \\ x_{21} & x_{22} & \cdots & x_{2(n-k)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & \cdots & x_{k(n-k)} \end{pmatrix} \quad \text{and} \quad \Delta' = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

We think of the entries x_{ij} of $X_{\mathcal{U}}$ as variables. The corresponding polynomial ring is the coordinate ring of $\mathcal{U} \simeq \mathbb{A}^{k(n-k)}$, and will be denoted by $\mathbb{C}[\mathcal{U}] = \mathbb{C}[x_{ij}]$.

\mathcal{U} and its Weyl translates form a natural system of affine charts for the projective variety $G(k, n)$, and there is a natural “de-homogenization process” from $\mathbb{C}[G(k, n)]$ to $\mathbb{C}[\mathcal{U}]$. Explicitly, this consists in substituting the matrix X with the matrix $(X_{\mathcal{U}} | \Delta')$.

We use the notation $[i_1 \dots i_r | j_1 \dots j_r]$ for the minor of $X_{\mathcal{U}}$ corresponding to the rows i_s and the columns j_s . By convention we set $[|] = 1$. Via the “de-homogenizing” substitution mentioned above (and ignoring signs), we obtain a bijection between Plücker coordinates $[i_1 \dots i_k]$ and minors $[i_1 \dots i_r | j_1 \dots j_r]$ (of arbitrary size) of $X_{\mathcal{U}}$. Some examples of this process are shown in Fig. 2C.

Plücker coordinate	Minor
[123]	$[123 123] = \det(x_{ij})_{i,j=1\dots 3}$
[124]	$[12 12] = x_{11}x_{22} - x_{12}x_{21}$
[235]	$[13 23] = x_{12}x_{33} - x_{13}x_{32}$
[246]	$[2 2] = x_{22}$
[456]	$[] = 1$

FIGURE 2C. Examples of de-homogenizations of Plücker coordinates in $G(3, 6)$. Signs have been ignored.

Using the bijection with Plücker coordinates we endow the set of minors with an order. If M_1, M_2 are minors, with corresponding Plücker coordinates I_1, I_2 , and corresponding

partitions λ_1, λ_2 , we have:

$$M_1 \leq M_2 \quad \Leftrightarrow \quad I_1 \leq I_2 \quad \Leftrightarrow \quad \lambda_1 \subseteq \lambda_2.$$

Explicitly, if $M_1 = [i_1 \dots i_r | j_1 \dots j_r]$ and $M_2 = [a_1 \dots a_s | b_1 \dots b_s]$, then:

$$M_1 \leq M_2 \quad \Leftrightarrow \quad \begin{array}{l} r \geq s \\ i_u \leq a_u \quad \text{for } 1 \leq u \leq s. \\ j_u \leq b_u \end{array} \quad (2a)$$

See the last diagram in Fig. 2B for the example of $G(2, 4)$.

All Schubert varieties intersect the opposite big cell, and the ideal of this intersection can be determined using Theorem 2.1. Let I be a multi-index, with corresponding minor M . Then the ideal of $\Omega_I \cap \mathcal{U}$ in the ring $\mathbb{C}[\mathcal{U}] = \mathbb{C}[x_{ij}]$ is generated by the minors M_1 such that $M_1 \not\geq M$.

Single Schubert conditions. We denote by $(b^a) = (b \cdot^a \cdot b)$ the rectangular partition with a rows and b columns. A *single-condition Schubert variety* is a Schubert variety Ω_λ whose associated partition λ has rectangular shape. In such case, if $\lambda = (b^a)$, we have

$$\Omega_{(b^a)} = \{V \in G(k, n) \mid \dim V \cap F_{n-k+a-b} \geq a\}.$$

Given an arbitrary partition λ , we define the *Schubert conditions* of λ to be the maximal rectangular partitions contained in λ . This definition is perhaps best illustrated by examples: see Fig. 2D.

Partition			
Schubert conditions			

FIGURE 2D. Schubert conditions

Proposition 2.3. *Let λ be a partition, and let μ_1, \dots, μ_r be the Schubert conditions of λ . Then*

$$\Omega_\lambda = \Omega_{\mu_1} \cap \dots \cap \Omega_{\mu_r} \quad \text{and} \quad \mathcal{I}_\lambda = \mathcal{I}_{\mu_1} + \dots + \mathcal{I}_{\mu_r}.$$

Proof. This is an immediate consequence of Theorem 2.1. Let $[j_1 \dots j_k]$ be a multi-index with associated partition μ . Then $[j_1 \dots j_k]$ is a generator of \mathcal{I}_λ if and only if $\mu \not\geq \lambda$. But $\mu \not\geq \lambda$ if and only if $\mu \not\geq \mu_s$ for some $1 \leq s \leq r$, and the result follows. \square

The ideals defining single-condition Schubert varieties have a particularly simple structure, especially in the opposite big cell. Given $1 \leq a \leq k$ and $1 \leq b \leq n - k$, we denote by $M_{a,b}$ the minor with the biggest size having first row a and first column b . It is easy to see that $M_{a,b} = [a \dots a + r | b \dots b + r]$, where $r = \min\{k - a, n - k - b\}$. We will refer to the minors obtained in this way as *final minors*.

Given a final minor $M_{a,b}$, we consider the corresponding multi-index $I_{a,b}$ and partition $\lambda_{a,b}$. They are given by

$$I_{a,b} = [(b) \dots (b + k - a) (n - a + 2) \dots (n)] \quad (2b)$$

and

$$\lambda_{a,b} = ((n - k) \dots (n - k) (b - 1) \dots (b - 1)),$$

where $(n - k)$ shows up $(a - 1)$ times in $\lambda_{a,b}$, and $(b - 1)$ shows up $(k - a + 1)$ times. More visually, $\lambda_{a,b}$ is the biggest partition not containing the box in position (a, b) .

Proposition 2.4. *Let $\lambda = (b^a)$ be a rectangular partition, and consider $I_{a,b}$ and $M_{a,b}$ as above. Let $r = \min\{k - a, n - k - b\}$, so that the size of $M_{a,b}$ is $r + 1$. Then:*

- (1) *The ideal $\mathcal{I}_\lambda \subset \mathbb{C}[G(k, n)]$ is generated by the Plücker coordinates J that verify $J \leq I_{a,b}$.*
- (2) *In the opposite big cell \mathcal{U} , the ideal of $\Omega_\lambda \cap \mathcal{U}$ is generated by the minors M that verify $M \leq M_{a,b}$. Moreover, it is enough to consider only minors M of size $r + 1$. More precisely, if $(k - a) \leq (n - k - b)$ the generators are the minors of size $r + 1$ in the first $b + r$ columns of $X_\mathcal{U}$. If $(k - a) \geq (n - k - b)$ the generators are the minors of size $r + 1$ in the first $a + r$ rows of $X_\mathcal{U}$.*

Proof. Let I be the Plücker coordinate corresponding to λ . By Theorem 2.1, the ideal \mathcal{I}_λ is generated by the Plücker coordinates J such that $J \not\leq I$. This is equivalent to $\mu \not\leq \lambda$, where μ is the partition associated to J . Since λ is rectangular, this happens precisely when μ does not contain the box (a, b) . By the discussion preceding the proposition, this is equivalent to $\mu \subseteq \lambda_{a,b}$, and the first part follows. The second part is an immediate consequence of the first part and the definition of the order among minors (Eq. (2a)). \square

3. A decomposition of the arc space of the Grassmannian

In this section we give a decomposition of the arc space of the Grassmannian that resembles the Schubert cell decomposition of the Grassmannian itself. We will call the pieces of this decomposition contact strata. Just as Schubert cells are indexed by partitions, contact strata are indexed by plane partitions.

Recall that we write $J_\infty G(k, n)$ and $J_m G(k, n)$ for the arc space and jet schemes of $G(k, n)$. The universal property of the Grassmannian [GD71, §9.7] tells us what the \mathbb{C} -valued points of $J_\infty G(k, n)$ are: they correspond to $\mathbb{C}[[t]]$ -submodules $\Lambda \subset \mathbb{C}[[t]]^n$ for which the corresponding quotient $\mathbb{C}[[t]]^n / \Lambda$ is free of rank $n - k$. Each such Λ is itself free, and of rank k , so it can be represented by a $k \times n$ matrix with coefficients in $\mathbb{C}[[t]]$:

$$\Lambda = \text{row span} \begin{pmatrix} x_{11}(t) & x_{12}(t) & \dots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \dots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1}(t) & x_{k2}(t) & \dots & x_{kn}(t) \end{pmatrix}, \quad x_{ij}(t) \in \mathbb{C}[[t]].$$

The condition on the freeness of the quotient $\mathbb{C}[[t]]^n / \Lambda$ simply says that one of the maximal minors of this matrix is a unit in $\mathbb{C}[[t]]$. Notice that the matrix is only determined by Λ up to multiplication on the right by an element of $\text{GL}_k(\mathbb{C}[[t]])$. Despite this, we will often use the same symbol Λ to also denote any matrix representing Λ .

There is an analogous description for \mathbb{C} -valued points of the jet schemes $J_m G(k, n)$.

Definition 3.1. Let Λ be an arc in $J_\infty G(k, n)$. The collection

$$\{\text{ord}_\Lambda(\Omega_\lambda)\}_\lambda,$$

where λ ranges among all partitions with $\leq k$ parts of size $\leq n - k$, is called the *contact profile* of Λ (with respect to the Schubert varieties). Given a collection

$$\alpha = \{\alpha_\lambda\}_\lambda,$$

where $\alpha_\lambda \in [0, \infty]$ and λ ranges as above, the *contact stratum* of $J_\infty G(k, n)$ associated to α is the set of arcs that have contact profile α . We define analogously contact profiles for jets in $J_m G(k, n)$, and contact strata in $J_m G(k, n)$.

Not all collections of numbers $\{\alpha_\lambda\}_\lambda$ appear as contact profiles of arcs and jets. The following results address the issue of enumerating all possible contact profiles. As we will see, contact profiles are in bijection with certain plane partitions.

Proposition 3.2. *Let $\{\alpha_\lambda\}_\lambda$ be the contact profile of an arc or a jet in $G(k, n)$. Fix a partition λ , and let μ_1, \dots, μ_r be the Schubert conditions of λ (as in Proposition 2.3). Then*

$$\alpha_\lambda = \min\{\alpha_{\mu_1}, \dots, \alpha_{\mu_r}\}.$$

Proof. This follows immediately from Proposition 2.3. \square

Definition 3.3. For positive integers i and j , recall that (j^i) denotes the rectangular partition with i rows and j columns. If Λ is an arc or a jet in $G(k, n)$, the matrix α of size $k \times (n - k)$ with entries

$$\alpha_{i,j} = \text{ord}_\Lambda(\Omega_{(j^i)})$$

is called the *essential contact profile* of Λ . Notice that Proposition 3.2 guarantees that the essential contact profile determines the contact profile.

Proposition 3.4. *Let $\alpha = (\alpha_{i,j})$ be the essential contact profile of an arc $G(k, n)$. Then*

$$\begin{aligned} \alpha_{i,j} &\geq \alpha_{i',j'} \quad \text{if } i \leq i' \text{ and } j \leq j' \\ \alpha_{i,j} + \alpha_{i+2,j+1} &\geq \alpha_{i+1,j} + \alpha_{i+1,j+1} \\ \alpha_{i,j} + \alpha_{i+1,j+2} &\geq \alpha_{i,j+1} + \alpha_{i+1,j+1} \end{aligned}$$

The same statements are true for essential contact profiles of m -jets, provided one replaces sums $x + y$ with $\min\{x + y, m + 1\}$.

Proof. The first inequality follows from the inclusions $\Omega_{(j^i)} \supseteq \Omega_{(j'^{i'})}$. We will prove the second inequality, the third one following from similar arguments.

Let Λ be an arc or a jet in $G(k, n)$ with essential contact profile $(\alpha_{i,j})$. The Borel group B has a right action on the arc space and jet schemes, and all the elements in the orbit $\Lambda \cdot B$ have the same contact profile.

Let $\mathcal{I}(i, j) = \mathcal{I}_{(j^i)}$ be the ideal of $\Omega_{(j^i)}$, and let $I_{i,j}$ be distinguished Plücker coordinate of $\mathcal{I}(i, j)$, as it appears in Proposition 2.4. The other generators J of $\mathcal{I}(i, j)$ verify $J \leq I_{i,j}$, that is, the columns that appear in J are to the left of the columns that appear in $I_{i,j}$. Notice that B acts by column operations, in such a way that columns on the left affect columns on the right. Therefore, after replacing Λ by a generic B -translate, we can assume that $\alpha_{i,j} = \text{ord}_\Lambda(I_{i,j})$ for all i, j . The proposition will be proven if we show that

$$I_{i,j} \cdot I_{i+2,j+1} \in \mathcal{I}(i+1, j) \mathcal{I}(i+1, j+1). \quad (3a)$$

From Eq. (2b) we see that:

$$\begin{aligned} I_{i,j} &= [j \quad j+1 \quad \dots \quad c-1 \quad c \quad \quad \quad d+2 \quad \dots \quad n], \\ I_{i+2,j+1} &= [\quad j+1 \quad \dots \quad c-1 \quad \quad d \quad d+1 \quad d+2 \quad \dots \quad n], \\ I_{i+1,j} &= [j \quad j+1 \quad \dots \quad c-1 \quad \quad d+1 \quad d+2 \quad \dots \quad n], \\ I_{i+1,j+1} &= [\quad j+1 \quad \dots \quad c-1 \quad c \quad d+1 \quad d+2 \quad \dots \quad n], \end{aligned}$$

where $c = j + k - i$ and $d = n - i$. We apply Theorem 2.2 to the product $I_{i,j} \cdot I_{i+2,j+1}$ using $u = k - i + 1$. We obtain an expansion

$$I_{i,j} \cdot I_{i+2,j+1} = \sum_v \pm I^v \cdot J^v,$$

where I^v and J^v are obtained from $I_{i,j}$ and $I_{i+2,j+1}$ by swapping the entry with value c in $I_{i,j}$ with the v -th entry in $I_{i+2,j+1}$. After the swap in most cases I^v is zero (it has a repeated entry). The only exceptions are $v = k - i$ and $v = k - i + 1$, that is, when c gets swapped with either d or $d + 1$. In both cases it is easy to see that

$$I^v \leq I_{i+1,j} \quad \text{and} \quad J^v \leq I_{i+1,j+1}.$$

Now Eq. (3a) follows from Proposition 2.4, and the proposition is proven. \square

Corollary 3.5. *Let Λ be an arc or a jet in $G(k, n)$, and let $\{\alpha_\lambda\}_\lambda$ and $(\alpha_{i,j})$ be the associated contact profile and essential contact profile. Then:*

$$\alpha_\lambda = \min\{\alpha_{i,j} \mid (i, j) \in \lambda\}.$$

Definition 3.6. Let Λ be an arc in $G(k, n)$, and let $(\alpha_{i,j})$ be its essential contact profile. Consider the matrix β of size $k \times (n - k)$ with entries

$$\beta_{i,j} = \alpha_{i,j} - \alpha_{i+1,j+1},$$

where we set $\alpha_{i',j'} = 0$ when $i' > k$ or $j' > n - k$, and use the convention $\infty - x = \infty$. Then β is called the *invariant factor profile* of Λ . This choice of terminology will be justified in Section 4. Notice that we only define invariant factor profiles for arcs, not jets. For an extension of this notion to jets, see Remark 4.2.

Remark 3.7 (Plane partitions). Recall that a *plane partition* (sometimes also called a *3d partition*) is a matrix of non-negative integers whose entries are non-increasing along each column and along each row. We slightly generalize this notion and *allow entries to be infinite*. Two plane partitions are identified when they have the same non-zero entries, and therefore they are often written by omitting the zero entries. The collection of non-zero entries in a plane partition β determines a (linear) partition λ , called the *base* (or *shape*) of the plane partition, and gives the *number of columns* and the *number of rows* of β . The biggest entry of β (the one in position $(1, 1)$) is called the *height* of β . The *number of boxes* in β (or the *sum* of β , or the *volume* of β) is the sum of the entries of β . A plane partition $\beta = (\beta_{i,j})$ is often visualized via its *Ferrers-Young diagram*, a collection of boxes in space, with a pillar of height $\beta_{i,j}$ on top of the square in the plane in position (i, j) . For some examples see Fig. 3A.

Theorem 3.8. *Let $\beta = (\beta_{i,j})$ be the invariant factor profile of an arc in $G(k, n)$. Then β is a plane partition (possibly with infinite height), i.e.,*

$$\begin{aligned} \beta_{i,j} &\geq 0, \\ \beta_{i,j} &\geq \beta_{i+1,j}, \\ \beta_{i,j} &\geq \beta_{i,j+1}. \end{aligned}$$

Conversely, any plane partition with base contained in the rectangle of size $k \times (n - k)$ is the invariant factor profile of some arc in $G(k, n)$.

The first part of the previous theorem follows immediately from Proposition 3.4. The “converse” part, the fact that all plane partitions give rise to a non-empty contact stratum is harder to prove, and requires some preparation. The proof appears in Section 6 as a consequence of Theorem 6.3.

Notation 3.9. Given a plane partition β (possibly with infinite height), we denote by \mathcal{C}_β the contact stratum in $J_\infty G(k, n)$ whose arcs have invariant factor profile equal to β . The previous theorem guarantees that all contact strata in $J_\infty G(k, n)$ are of the form \mathcal{C}_β for some β , and that \mathcal{C}_β is non-empty precisely when β has its base contained in the rectangle of size $k \times (n - k)$.

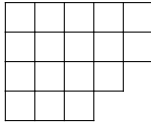
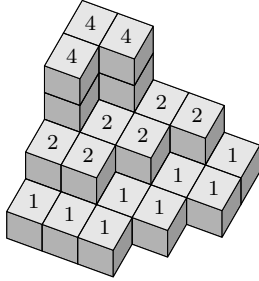
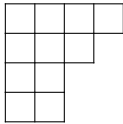
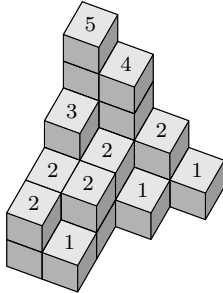
Plane Partition	Base	Diagram
$ \begin{array}{cccc} 4 & 4 & 2 & 2 & 1 \\ 4 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 & \\ 1 & 1 & 1 & & \end{array} $	(5543) 	
$ \begin{array}{cccc} 5 & 4 & 2 & 1 \\ 3 & 2 & 1 & \\ 2 & 2 & & \\ 2 & 1 & & \end{array} $	(4322) 	

FIGURE 3A. Some examples of plane partitions and their diagrams.

4. Contact strata and Schubert conditions

Before finishing the proof of Theorem 3.8, we discuss another interpretation for the numbers that appear in the invariant factor profile of an arc. This interpretation justifies our terminology.

The idea is to study “Schubert conditions for lattices”. We start by recalling how to do this for Schubert cells. As before, we consider the flag F_\bullet given by

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n,$$

where F_i is spanned by the last i vectors in the standard basis of \mathbb{C}^n . Then, given a point $V \in G(k, n)$, the numbers

$$\rho_s = \dim_{\mathbb{C}}(V \cap F_s)$$

determine, and are determined by, the Schubert cell Ω_I° to which V belongs. Indeed, the ρ_s are clearly invariant under B -translates, so we can assume that $V = V_I = \langle e_{i_1}, \dots, e_{i_k} \rangle$. In this case it is easy to see that $k - \rho_s = \dim_{\mathbb{C}}(V_I + F_s/F_s)$ is the number of entries in I less than or equal to $n - s$.

We would like to characterize contact strata \mathcal{C}_β in an analogous way. But in order to do this, the above description using intersections $V \cap F_\bullet$ is not convenient. It is better to consider the quotients $\mathbb{C}^n/(V + F_\bullet)$. More precisely, for a point $V \in \Omega_I^\circ$ we consider the partition $\mu = (\mu_0 \dots \mu_n)$ given by

$$\dim_{\mathbb{C}} \left(\frac{\mathbb{C}^n}{V + F_s} \right) = \mu_s \quad (4a)$$

for $0 \leq s \leq n$. Then $n - s - \mu_s = \dim_{\mathbb{C}}(V + F_s/F_s) = k - \rho_s$, and therefore the μ_s are determined by I (and vice-versa). A direct computation shows that μ_s can be computed more visually in the following way. We consider the diagram of the partition λ associated to I . Above this diagram we place, upside-down, the diagram of $(n - k \dots 1)$. Then the entries in μ count the number of boxes in the diagonals of the resulting arrangement of boxes. For examples of this computation see Fig. 4A.

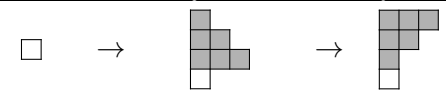
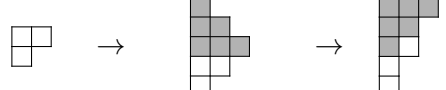
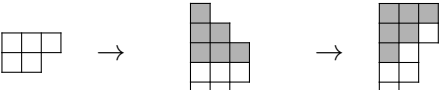
I	λ	$\lambda \rightarrow (\begin{smallmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{smallmatrix} + \lambda) \rightarrow \mu$	μ
[13]	(1)		(321100)
[24]	(21)		(322110)
[35]	(32)		(332210)

FIGURE 4A. Computing μ in $G(2, 5)$.

We interpret the above equations on dimensions as statements about the isomorphism type of vector spaces. The elements of the arc space $J_{\infty}G(k, n)$ are lattices, and the dimension (or rank) is no longer a complete invariant of the isomorphism type of a $\mathbb{C}[[t]]$ -module. Instead, using the structure theory for finitely generated modules over a PID, we know that every finitely generated $\mathbb{C}[[t]]$ -module Γ has a unique expression of the form

$$\Gamma \simeq \frac{\mathbb{C}[[t]]}{t^{\lambda_1}} \oplus \frac{\mathbb{C}[[t]]}{t^{\lambda_2}} \oplus \dots \oplus \frac{\mathbb{C}[[t]]}{t^{\lambda_m}}$$

where $\lambda_i \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$. As usual, we are using the convention $t^{\infty} = 0$, so the free rank of Γ is the number of infinite terms among the λ_i . The numbers λ_i are called the *invariant factors* of Γ .

With a slight abuse of notation, we also denote by F_{\bullet} the flag

$$0 = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_n = \mathbb{C}[[t]]^n,$$

where F_i is the $\mathbb{C}[[t]]$ -span of the last i vectors in the standard basis of $\mathbb{C}[[t]]^n$.

Theorem 4.1. *Let $\beta = (\beta_{i,j})$ be a plane partition (possibly with infinite height) whose base is contained in the rectangle of size $k \times (n - k)$. Extend β by setting $\beta_{i,j} = \infty$ for $i < 1$ and $\beta_{i,j} = 0$ for $i > k$. Let Λ be an arc in $G(k, n)$, thought as a $\mathbb{C}[[t]]$ -submodule of $\mathbb{C}[[t]]^n$. Then Λ belongs to the contact stratum \mathcal{C}_{β} if and only if the quotient module*

$$\frac{\mathbb{C}[[t]]^n}{\Lambda + F_i}$$

has invariant factors

$$\beta_{i+k-n+1,1} \quad \beta_{i+k-n+2,2} \quad \dots \quad \beta_{i-1,n-k-1} \quad \beta_{i,n-k}$$

for $1 \leq i \leq n - 1$.

See Fig. 0A for a diagram explaining the content of Theorem 4.1 in the particular case of $G(2, 5)$. Notice how it is a natural generalization of the description of Schubert cells given in Eq. (4a).

Proof. The action of the Borel group B on \mathbb{C}^n naturally induces an action on $\mathbb{C}[[t]]^n$, and the flag F_\bullet is fixed by B . In particular, for any element $b \in B$ we obtain isomorphisms

$$\frac{\mathbb{C}[[t]]^n}{\Lambda + F_i} \simeq \frac{\mathbb{C}[[t]]^n}{(\Lambda \cdot b) + F_i}$$

for all i . Also, both Λ and $\Lambda \cdot b$ have the same contact profile, and therefore the same invariant factor profile. Because of these facts, in order to prove the theorem we are free to replace Λ with any of its B -translates.

Replace Λ with a generic B -translate. This implies that Λ is contained in the opposite big cell, that is, it can be written as the row span of a matrix Λ_0 of the form

$$\Lambda_0 = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1(n-k)} & 0 & \cdots & 0 & 1 \\ x_{21} & x_{22} & \cdots & x_{2(n-k)} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{k1} & x_{k2} & \cdots & x_{k(n-k)} & 1 & \cdots & 0 & 0 \end{pmatrix}, \quad x_{ij} \in \mathbb{C}[[t]].$$

We let Λ_i be the matrix of size $k \times (n-i)$ obtained by removing the last i columns of Λ_0 . After identifying $\mathbb{C}[[t]]^n/F_i$ with $\mathbb{C}[[t]]^{n-i}$, the submodule $\Lambda + F_i/F_i$ can be obtained as the row span of Λ_i . In particular, the invariant factors of $\mathbb{C}[[t]]^n/\Lambda + F_i$ can be computed by looking at ideals of minors of Λ_i .

More explicitly, for $1 \leq j \leq n-i$, let $\mathfrak{d}_{i,j}$ be the ideal generated by the minors of size $n+1-i-j$ of Λ_i . Here we set a minor equal to 0 if its size is bigger than the size of Λ_i . We know that $\mathfrak{d}_{i,j}$ is of the form $(t^{d_{i,j}})$ for some $d_{i,j} \in [0, \infty]$. Then the invariant factors of $\mathbb{C}[[t]]^n/\Lambda + F_i$ are:

$$d_{i,1} - d_{i,2} \quad d_{i,2} - d_{i,3} \quad \cdots \quad d_{i,n-i-1} - d_{i,n-i} \quad d_{i,n-i}$$

Let $(\alpha_{i,j})$ be the essential contact profile of Λ , extended by setting $\alpha_{i,j} = \infty$ if $i < 1$ and $\alpha_{i,j} = 0$ if $i > k$ or $j > n-k$. From the previous discussion we see that the theorem follows if we prove that

$$\alpha_{i+k-n+j,j} = d_{i,j}.$$

Observe that when $k < n+1-i-j$ both $\alpha_{i+k-n+j,j}$ and $d_{i,j}$ are equal to ∞ . So we can assume that $i+k-n+j \geq 1$. Also, when $i < k$ the matrix Λ_i has minors of any size $\leq k-i$ that are equal to 1. Therefore, if $i < k$ and $j > n-k$ we see that $\mathfrak{d}_{i,j}$ must be the unit ideal, since it contains all minors of size $n+1-i-j < k+1-i$. In particular in this case $\alpha_{i+k-n+j,j} = d_{i,j} = 0$. Since we always have $j \leq n-i$, we can assume that $j \leq n-k$. Finally, also using that $j \leq n-i$, we see that $i+k-n+j \leq k$.

We use Proposition 2.4 with $a = i+k-n+j$ and $b = j$. Notice that the previous paragraph guarantees that we can consider $1 \leq a \leq k$ and $1 \leq b \leq n-k$. Let $\mathfrak{a}_{a,b}$ be the ideal of $\Omega_{(b^a)} \cap \mathcal{U}$. We will show that $\mathfrak{a}_{a,b} = \mathfrak{d}_{i,j}$. Consider

$$r = \min\{k-a, n-k-b\} = \min\{n-i-j, n-k-j\} > 0,$$

and recall that $\mathfrak{a}_{a,b}$ is generated by certain minors of size $r+1$.

According to Proposition 2.4, we have two cases. If $i \geq k$, then $r = n-i-j$ and $\mathfrak{a}_{a,b}$ is generated by the minors of size $r+1$ in the first $b+r$ columns. In this case $b+r = n-i$, and we see that $\mathfrak{a}_{a,b} = \mathfrak{d}_{i,j}$.

The other possibility is $i < k$, which implies $r = n-k-j$. We can write Λ_i in block form:

$$\Lambda_i = \left(\begin{array}{c|c} A_i & 0 \\ \hline B_i & C_i \end{array} \right),$$

where C_i is the anti-diagonal matrix of size $(k-i) \times (k-i)$. Then $\mathfrak{a}_{a,b}$ is generated by the minors of A_i of size $n+1-k-j$, while $\mathfrak{d}_{i,j}$ contains all minors of Λ_i of size $n+1-i-j$. Since determinantal ideals are independent of the choice of basis, we can perform column operations on Λ_i until we obtain a matrix of the form

$$\tilde{\Lambda}_i = \left(\begin{array}{c|c} A_i & 0 \\ \hline 0 & C_i \end{array} \right),$$

and $\mathfrak{d}_{i,j}$ is still the ideal generated by all minors of $\tilde{\Lambda}_i$ of size $n+1-i-j$. But from the above block form we see that $\mathfrak{d}_{i,j}$ must be generated by the minors of $\tilde{\Lambda}_i$ of size $n+1-i-j$ which contain the last $k-i$ columns, and these are clearly equal to the minors of A_i of size $n+1-k-j$. In other words, $\mathfrak{a}_{a,b} = \mathfrak{d}_{i,j}$, and the theorem follows. \square

Remark 4.2 (Invariant factor profiles for jets). We can use Theorem 4.1 to extend the definition of invariant factor profile to jets. Given a jet $\Lambda \in J_m G(k, n)$ we define the matrix $\beta = (\beta_{i,j})$ in such a way that each quotient module

$$\frac{\mathbb{C}[t]/(t^{m+1})}{\Lambda + F_i}$$

has invariant factors

$$\beta_{i+k-n+1,1} \quad \beta_{i+k-n+2,2} \quad \cdots \quad \beta_{i-1,n-k-1} \quad \beta_{i,n-k}.$$

Then β is called the *invariant factor profile* of Λ . The same ideas of the proof of Theorem 4.1 show that the essential contact profile α of Λ can be recovered from the invariant factor profile:

$$\alpha_{i,j} = \beta_{i,j} + \beta_{i+1,j+1} + \cdots.$$

But notice that, in contrast with what happens for arcs, the invariant factor profile of a jet cannot be recovered in general from its contact profile.

We still denote by \mathcal{C}_β the set of jets with invariant factor profile β . Each contact stratum is a union of the \mathcal{C}_β .

5. Orbits in the arc space

Theorem 4.1 provides a strong motivation for considering contact strata, but there are other natural decompositions of the arc space of the Grassmannian, mainly coming from groups actions.

Recall that the action of GL_n on $G(k, n)$ induces an action at the level of arc spaces: $J_\infty \mathrm{GL}_n$ acts on $J_\infty G(k, n)$. The arc space $J_\infty \mathrm{GL}_n$ is the group of invertible matrices with coefficients in $\mathbb{C}[[t]]$, and its action on $J_\infty G(k, n)$ is by column operations (also with coefficients in $\mathbb{C}[[t]]$). This identifies the arc space of the Grassmannian with the homogeneous space

$$J_\infty G(k, n) = \frac{J_\infty \mathrm{GL}_n}{J_\infty P_{k,n}} = \frac{\mathrm{GL}_n(\mathbb{C}[[t]])}{P_{k,n}(\mathbb{C}[[t]])},$$

where $P_{k,n} \subset \mathrm{GL}_n$ is the parabolic subgroup described in Section 2.

The above presentation of the arc space of the Grassmannian as a homogeneous space should not be confused with other quotients of similar type that appear frequently in the literature. The affine Grassmannian and the affine Flag variety (in type A) are defined as

$$\mathcal{G}r_n = \frac{\mathrm{GL}_n(\mathbb{C}((t)))}{\mathrm{GL}_n(\mathbb{C}[[t]])} \quad \text{and} \quad \mathcal{F}l_n = \frac{\mathrm{GL}_n(\mathbb{C}((t)))}{\mathcal{B}_n},$$

where $\mathcal{B}_n = \text{Cont}^{\geq 1}(B) \subset J_\infty \text{GL}_n$ is the Iwahori group. Denote by $Fl(n) = B \backslash \text{GL}_n$ the Flag manifold. Then there are natural fibrations relating the different quotients. We have

$$\begin{array}{ccc} \frac{\text{GL}_n(\mathbb{C}((t)))}{B(\mathbb{C}[[t]])} & \xrightarrow{\quad} & \mathcal{F}l_n \xrightarrow{Fl(n)} \mathcal{G}r_n \\ & \searrow & \uparrow \\ & & J_\infty Fl(n) \end{array}$$

for flags, and

$$\begin{array}{ccc} \frac{\text{GL}_n(\mathbb{C}((t)))}{P_{k,n}(\mathbb{C}[[t]])} & \xrightarrow{\quad} & \frac{\text{GL}_n(\mathbb{C}((t)))}{\text{Cont}^{\geq 1}(P_{k,n})} \xrightarrow{G(k,n)} \mathcal{G}r_n \\ & \searrow & \uparrow \\ & & J_\infty G(k,n) \end{array}$$

for the Grassmannian. In the above two diagrams, the objects labeling the arrows represent the fibers of the corresponding fibrations. Observe that, from this point of view, the arc spaces $J_\infty G(k, n)$ and $J_\infty Fl(n)$ are very different from $\mathcal{G}r_n$ and $\mathcal{F}l_n$. The “affine” objects are well behaved representation theoretically: they are natural homogeneous spaces associated to a Kac-Moody group. On the other hand, the group associated to the arc spaces is $J_\infty \text{GL}_n$, which is isomorphic to the product of a reductive group, GL_n , with an infinite dimensional solvable group, $J_\infty \mathfrak{gl}_n = \text{Cont}^{\geq 1}(\text{Id}) \subset J_\infty \text{GL}_n$.

In principle, two subgroups of $J_\infty \text{GL}_n$ should be relevant from the point of view of the Grassmannian: the arc space of the Borel $J_\infty B$, and the Iwahori subgroup $\mathcal{B}_n = \text{Cont}^{\geq 1}(B) \subset J_\infty \text{GL}_n$. It is natural to consider the decomposition of $J_\infty G(k, n)$ into the orbits of either of these two groups. Perhaps surprisingly, and in contrast with what happens in $G(k, n)$, neither of these actions gives rise to contact strata.

The Iwahori orbits are just $\text{Cont}^{\geq 1}(\Omega_I^\circ)$, the inverse images of the Schubert cells. They provide little insight into the structure of $J_\infty G(k, n)$.

The orbits for $J_\infty B$ are more interesting, but we found them to be less apt for our study than contact strata. The main difficulty is the lack of a good combinatorial device parametrizing all the orbits. Notice that contact strata are invariant under the action of $J_\infty B$, and therefore they are unions of orbits. But there are a lot more orbits than contact strata.

Even though we will not pursue a full study of the $J_\infty B$ -orbits, we would like to give an idea of their complexity. To simplify the discussion, we restrict ourselves to $G(2, 4)$, and we only consider arcs centered on the Borel fixed point. Such arcs are contained in the opposite big cell, and are represented by matrices of the form

$$\Lambda = \begin{pmatrix} x_{11} & x_{12} & 0 & 1 \\ x_{21} & x_{22} & 1 & 0 \end{pmatrix},$$

where the coefficients are in the maximal ideal, $x_{ij} \in (t) \subset \mathbb{C}[[t]]$. These arcs are characterized by having an invariant factor profile with base the full rectangle of size 2×2 . Given an element $b \in J_\infty B$, the translate $\Lambda \cdot b$ is an arc of the same type. In fact, there exists a unique $g \in J_\infty \text{GL}_2$ such that $\Lambda_b = g \cdot \Lambda \cdot b$ is again a matrix of the same form as above.

The group $J_\infty B$ is generated by the arc spaces of the torus, $J_\infty T$, and of the three unipotent subgroups corresponding to the positive roots, $J_\infty U_{12}$, $J_\infty U_{23}$, and $J_\infty U_{34}$. After straightforward computations, we can determine the explicit effect of the action on the matrix Λ for each of these generators. The results are summarized in Fig. 5A,

where we have used a particular torus twist of U_{23} (denoted \tilde{U}_{23}) so that the expression is more readable.

Original matrix	$\begin{pmatrix} x_{11} & x_{12} & 0 & 1 \\ x_{21} & x_{22} & 1 & 0 \end{pmatrix}$
Action of the Torus	$\begin{pmatrix} u_1 v_1 x_{11} & u_1 v_2 x_{12} & 0 & 1 \\ u_2 v_1 x_{21} & u_2 v_2 x_{22} & 1 & 0 \end{pmatrix}$
U_{12} – Column operation	$\begin{pmatrix} x_{11} & x_{12} + u x_{11} & 0 & 1 \\ x_{21} & x_{22} + u x_{21} & 1 & 0 \end{pmatrix}$
U_{34} – Row operation	$\begin{pmatrix} x_{11} & x_{12} & 0 & 1 \\ x_{21} + u x_{11} & x_{22} + u x_{12} & 1 & 0 \end{pmatrix}$
\tilde{U}_{23} ($\det = x_{11}x_{22} - x_{12}x_{21}$)	$\begin{pmatrix} x_{11} + u \det & x_{12} & 0 & 1 \\ x_{21} & x_{22} & 1 & 0 \end{pmatrix}$

FIGURE 5A. The generators of the action of $J_\infty B$ on $J_\infty G(2, 4)$ over the Borel fixed point. In the table u_i and v_i are units in $\mathbb{C}[[t]]$, and u is an arbitrary power series.

After these remarks, it is easy to construct matrices in different orbits but in the same contact stratum. An example is given by the arcs

$$\Lambda_1 = \begin{pmatrix} 0 & t^2 & 0 & 1 \\ t^2 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \Lambda_2 = \begin{pmatrix} t^3 & t^2 & 0 & 1 \\ t^2 & 0 & 1 & 0 \end{pmatrix},$$

whose invariant factor profile is

$$\beta = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

The moves in Fig. 5A do not allow to transform Λ_1 into Λ_2 . In order to change the 0 in position (1, 1) with a t^3 , the only operation that could help would be \tilde{U}_{23} . But the determinant \det , even after any other move, is a multiple of t^4 , so we can never obtain the desired t^3 .

Other examples display even more pathological behavior. We consider the matrices

$$\Lambda_u = \begin{pmatrix} ut^3 & t^2 & 0 & 1 \\ t^2 & t & 1 & 0 \end{pmatrix}, \quad u \in \mathbb{C} \setminus \{0, 1\},$$

which clearly belong to the same contact stratum. We will show that they give rise to different orbits. Notice that in this way we obtain a continuous family of orbits, whereas the family of contact strata is countable.

To see that no two among the Λ_u belong to the same orbit, we could argue as above, using the generators of $J_\infty B$. More directly, we proceed as follows. Consider

$$\Lambda_v = g \cdot \Lambda_u \cdot b, \tag{5a}$$

where $g \in J_\infty \text{GL}_2$ and $b = (b_{ij}) \in J_\infty B$. We apply all the Plücker coordinates $[ij]$ to both sides of Eq. (5a). After explicit computations we get:

$$\begin{aligned}
 [12]: \quad t^4(v-1) &= t^4(\det g) b_{11} b_{22} (u-1) \\
 [13]: \quad t^3 v &\equiv t^3(\det g) b_{11} b_{33} u \pmod{t^4} \\
 [23]: \quad t^2 &\equiv t^2(\det g) b_{22} b_{33} \pmod{t^3} \\
 [14]: \quad -t^2 &\equiv -t^2(\det g) b_{11} b_{44} \pmod{t^3} \\
 [24]: \quad -t &\equiv -t(\det g) b_{22} b_{44} \pmod{t^2} \\
 [34]: \quad -1 &\equiv -(\det g) b_{33} b_{44} \pmod{t}
 \end{aligned}$$

Focusing on the terms of lowest degree, the above equations imply:

$$\begin{aligned}
 b_{11}^0 b_{22}^0 \delta &= \frac{v-1}{u-1}, & b_{22}^0 b_{33}^0 \delta &= 1, & b_{22}^0 b_{44}^0 \delta &= 1, \\
 b_{11}^0 b_{33}^0 \delta &= \frac{v}{u}, & b_{11}^0 b_{44}^0 \delta &= 1, & b_{33}^0 b_{44}^0 \delta &= 1,
 \end{aligned}$$

where $b_{ii}^0 \neq 0$ is the constant coefficient of b_{ii} , and $\delta \neq 0$ is the constant coefficient of $\det g$. But these equations imply that $u = v$, as required.

We know very little about the $J_\infty B$ -orbit decomposition of $J_\infty G(k, n)$ in general. It would be interesting to understand these orbits better, and to study contact strata from this point of view.

6. Constructing arcs using planar networks

The goal of this section is to finish the proof of Theorem 3.8, that is, we want to show that all plane partitions appear as the invariant factor profile of an arc in the Grassmannian. In order to do this, we need to find a way of producing arcs with prescribed order of contact with respect to all the (single-condition) Schubert variates. This task seems to be highly non-trivial. We have borrowed ideas from the theory of the totally positive Grassmannian: we use planar networks to construct matrices, and use Lindström's Lemma to control the behavior of the minors. Most of the results in this section are adaptations to the case of arcs of the ideas in [FZ00].

Planar networks. A *planar network* Γ is an acyclic directed finite graph with a fixed embedding in the closed disk. We allow multiple edges, but no loops. We identify planar networks when they are homotopy equivalent (respecting the boundary of the disk). The vertices of Γ can be naturally classified into four types: *sources*, *sinks*, *internal vertices*, and *isolated vertices*. In all the cases that we consider, Γ has no isolated vertices, and can be drawn inside of the disc in such a way that the set of *boundary vertices* is the union of the sources and the sinks. Furthermore, Γ will have k sources, labeled from 1 to k clockwise along the boundary, and $(n - k)$ sinks, labeled counterclockwise.

When discussing *paths* on a planar network, we always assume that they are directed. A path is called *maximal* if it connects a source with a sink. A collection of paths is said to be *non-intersecting* if no two paths in the collection share a vertex.

A *chamber* of a planar network Γ is a connected component of the complement of Γ in the closed disk. Let p be a maximal path in Γ . Then p splits the disk in two connected components, which, taking into account the natural orientation of p , are called the *left* and *right* sides of p . Every chamber of Γ is either to the left or to the right of p .

A *weighting* of a planar graph Γ is a collection $w = \{w_v, w_e\}$, where v ranges among the internal vertices of Γ and e ranges among the edges of Γ . The elements of w belong to some ring fixed in advance, which in our case it will always be $\mathbb{C}[[t]]$. Using such a weighting w , we define the *weight of a path* p in Γ as the product of the weights of all the vertices and all the edges in p ,

$$w_p = \left(\prod_{v \in p} w_v \right) \left(\prod_{e \in p} w_e \right).$$

The *weight of a collection of paths* is the product of the weights of the paths in the collection.

The *weight matrix* $X(\Gamma, w)$ is the matrix of size $k \times (n - k)$ whose entry in position (i, j) is the sum of the weights of all (maximal) paths with source i and sink j ,

$$x_{ij} = \sum_{p \in \text{Paths}(i \rightarrow j)} w_p.$$

The following result makes calculations with weight matrices particularly convenient.

Lemma 6.1 (Lindström Lemma). *Let $X = X(\Gamma, w)$ be the weight matrix of a weighted planar network, and let $[I|J]$ be the minor of X with row set I and column set J . Then $[I|J]$ is the sum of the weights of the collections of non-intersecting paths that connect the sources in I with the sinks in J .*

For the proof we refer the reader to [FZ00, Lemma 1]. There it can be found the proof for the case where only edge weightings are used, but the same idea works for arbitrary weightings.

A particular network. Given a plane partition β , we will produce arcs in the contact stratum \mathcal{C}_β using a particular weight matrix. We now describe the corresponding planar network. The cases of $G(3, 8)$ (with $k = 3$ and $n - k = 5$) and $G(2, 5)$ (with $k = 2$ and $n - k = 3$) are given in Fig. 6A.

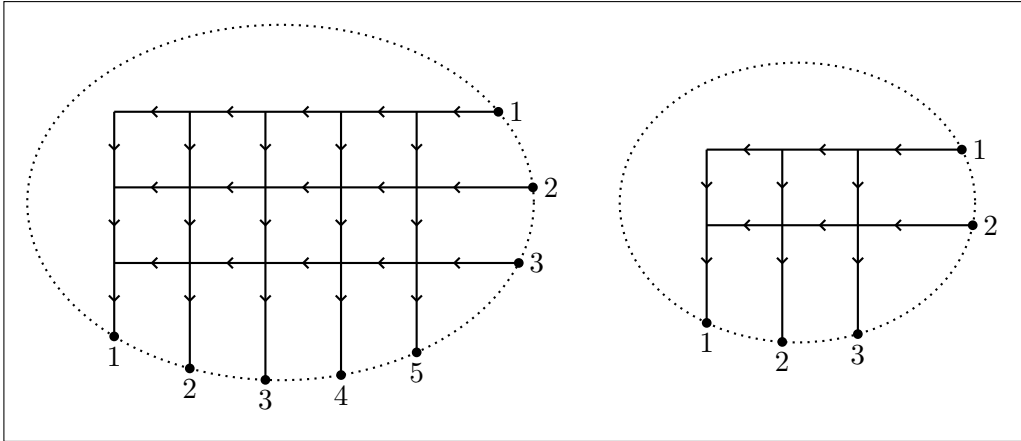


FIGURE 6A. The network Γ_0 .

The network we describe is denoted Γ_0 . It has k sources, $n - k$ sinks, and $k \times (n - k)$ internal vertices. The internal vertices are arranged using k rows and $(n - k)$ columns, and Γ_0 has edges connecting the internal edges to form a grid. The resulting k horizontal lines are extended towards the right until the boundary of the disk, where we place the

k sources. The $n - k$ vertical lines are extended towards the bottom, until the $n - k$ sinks. The horizontal edges are oriented from right to left, and the vertical ones from top to bottom. We get $k(n - k) + 1$ chambers, one on top, and the rest arranged in a grid of size $k \times (n - k)$. We label these last chambers C_{ij} , using matrix indexing.

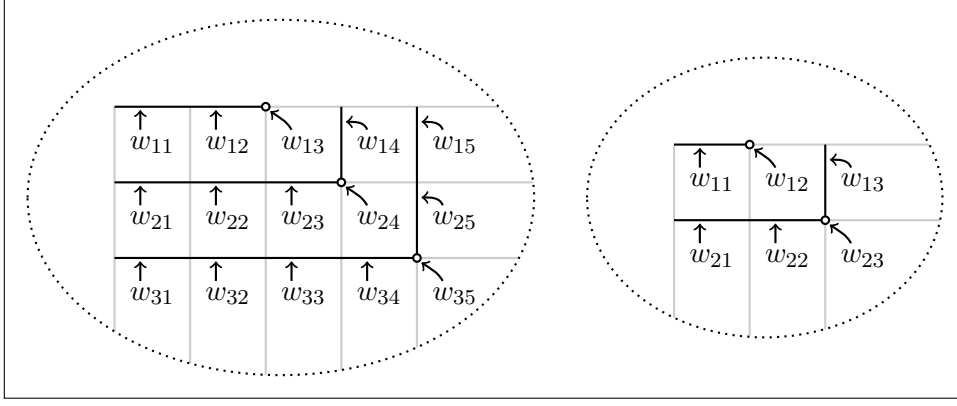


FIGURE 6B. The weights of Γ_0 . Only the dark edges and the marked vertices get a weight different from 1.

To assign weights to Γ_0 , we consider a matrix (w_{ij}) of size $k \times (n - k)$. If $k - i < n - k - j$ (resp. $k - i > n - k - j$), we assign weight w_{ij} to the edge on the top (resp. to the left) of the chamber C_{ij} . If $k - i = n - k - j$, we assign weight w_{ij} to the vertex in the top-left corner of C_{ij} . All other edges and vertices get weight 1. See Fig. 6B for some examples. Following [FZ00], we call a weighting w of Γ_0 obtained this way an *essential weighting*.

To compute the weight matrix $X(\Gamma_0, w)$, only the marked edges and vertices in Fig. 6B contribute some weight. For example, in $G(2, 5)$ we obtain:

$$X(\Gamma_0, w) = \begin{pmatrix} w_{12}w_{11} + w_{12}w_{21} + w_{13}w_{23}w_{22}w_{21} & w_{12} + w_{13}w_{23}w_{22} & w_{13}w_{23} \\ w_{23}w_{22}w_{21} & w_{23}w_{22} & w_{23} \end{pmatrix}.$$

We remark that our network Γ_0 is slightly different from the one used in [FZ00, Figure 2]. The reason is that the authors of [FZ00] prefer to use only edge weightings, while we decided to allow arbitrary weightings. The network in [FZ00] can be obtained from ours by replacing each weighted vertex with a diagonal edge (oriented from top-right to bottom-left), and moving weight from the vertices to these new edges. This process is explained visually in Fig. 6C.

Final minors in the weight matrix. From now on we assume that the weights w_{ij} belong to a power series ring, either $\mathbb{C}[[t]]$ or $K[[t]]$, where K is some field. We can think of $X(\Gamma_0, w)$ as giving a (K -valued) arc in the Grassmannian. More precisely, we consider

$$\Lambda(\Gamma_0, w) = \left(X(\Gamma_0, w) \mid \Delta' \right), \quad \text{where} \quad \Delta' = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Notice that $\Lambda(\Gamma_0, w)$ is an arc in the opposite big cell \mathcal{U} .

We are interested in understanding the invariant factor profile of $\Lambda(\Gamma_0, w)$. From Proposition 2.4, this involves studying the order of t in all the minors of $X(\Gamma_0, w)$. We start with the final minors (recall our terminology from Section 2).

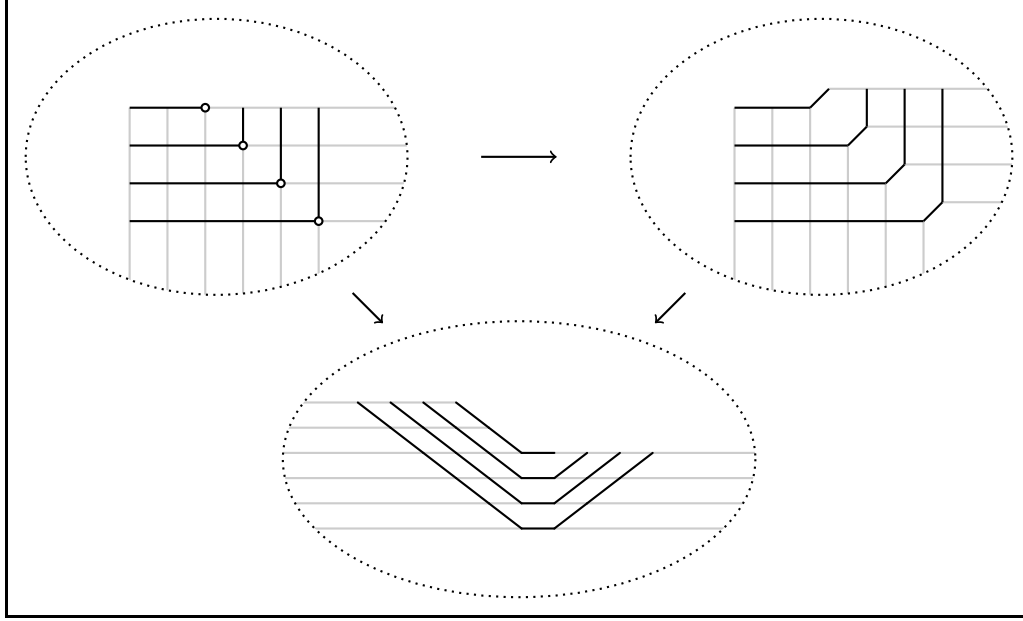


FIGURE 6C. The relationship between our network and the one in [FZ00].

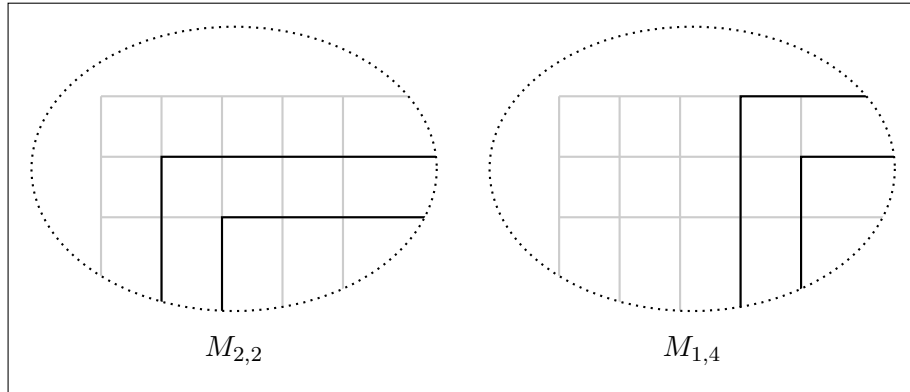
Lemma 6.2. *Let $M_{i,j}$ be a final minor of $X(\Gamma_0, w)$. Then there is a unique collection of non-intersecting paths of Γ_0 whose weight gives $M_{i,j}$. Moreover, if $(k-i) \leq (n-k-j)$ we have*

$$M_{i,j} = \prod_{u=0}^{k-i} \prod_{v=0}^{v_{\max}} w_{i+u, j+u+v}, \quad v_{\max} = (n-k-j) - (k-i),$$

and if $(k-i) \geq (n-k-j)$ we have

$$M_{i,j} = \prod_{u=0}^{u_{\max}} \prod_{v=0}^{n-k-j} w_{i+u+v, j+v}, \quad u_{\max} = (k-i) - (n-k-j).$$

The unique collection of paths is described in the proof. For an example, see Fig. 6D.

FIGURE 6D. The unique collections of paths in Γ_0 realizing the final minors $M_{2,2}$ and $M_{1,4}$ in $G(3, 8)$.

Proof. We use Lemma 6.1 to compute $M_{a,b}$. Recall that $M_{a,b} = [a \dots a+r | b \dots b+r]$, where $r = \min\{k-a, n-k-b\}$. In particular, $M_{a,b}$ involves either the last $r+1$ rows (if $b \leq a+n-2k$) or the last $r+1$ columns (if $b \geq a+n-2k$). In both cases there is a unique collection of non-intersecting paths of Γ_0 connecting the sources $\{a, \dots, a+r\}$ with the sinks $\{b, \dots, b+r\}$.

For example, assume $b \leq a+n-2k$. We need to connect $\{a, \dots, k\}$ with $\{b, \dots, b+r\}$. There is a unique path p_r in Γ_0 connecting the source k with the sink $b+r$. After removing p_r from Γ_0 , there is a unique path p_{r-1} from $k-1$ to $b+r-1$. Inductively, we produce the unique collection of paths $\{p_0, \dots, p_r\}$. To the weight of p_r there are only contributions from one marked vertex (the one in position $(k, n-k)$), and the marked horizontal edges from $(k, n-k)$ to $(k, b+r)$. We get that $w_{p_r} = \prod_{s=b+r}^{n-k} w_{k,s} = \prod_{v=0}^{v_{\max}} w_{k,b+r+v}$. Similarly $w_{p_u} = \prod_{v=0}^{v_{\max}} w_{a+u,b+u+v}$, and the formula for $M_{a,b}$ follows.

The case $b \geq a+n-2k$ is completely analogous. \square

Weight exponents. Given a plane partition $\beta = (\beta_{i,j})$ we define the *weight exponents* associated to β as the matrix $c(\beta) = (c_{i,j})$ given by:

$$\begin{aligned} c_{i,j} &= \beta_{i,j} & \text{if } (k-i) &= (n-k-j), \\ c_{i,j} &= \beta_{i,j} - \beta_{i,j+1} & \text{if } (k-i) < (n-k-j), \\ c_{i,j} &= \beta_{i,j} - \beta_{i+1,j} & \text{if } (k-i) > (n-k-j). \end{aligned}$$

Here we use the convention that $\infty - x = \infty$. For example, in the case of $G(3,6)$ we have:

$$c(\beta) = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} - \beta_{2,2} & \beta_{1,3} - \beta_{2,3} \\ \beta_{2,1} - \beta_{2,2} & \beta_{2,2} & \beta_{2,3} - \beta_{3,3} \\ \beta_{3,1} - \beta_{3,2} & \beta_{3,2} - \beta_{3,3} & \beta_{3,3} \end{pmatrix}$$

For a given essential weighting w of Γ_0 associated to a matrix $(w_{i,j})$ whose coefficients are power series, we define the *weight exponents* of w as the matrix $c(w) = (c_{i,j})$ given by

$$c_{i,j} = \text{ord}_t(w_{i,j}).$$

The main goal of this section is to prove the following theorem, which immediately concludes the proof of Theorem 3.8.

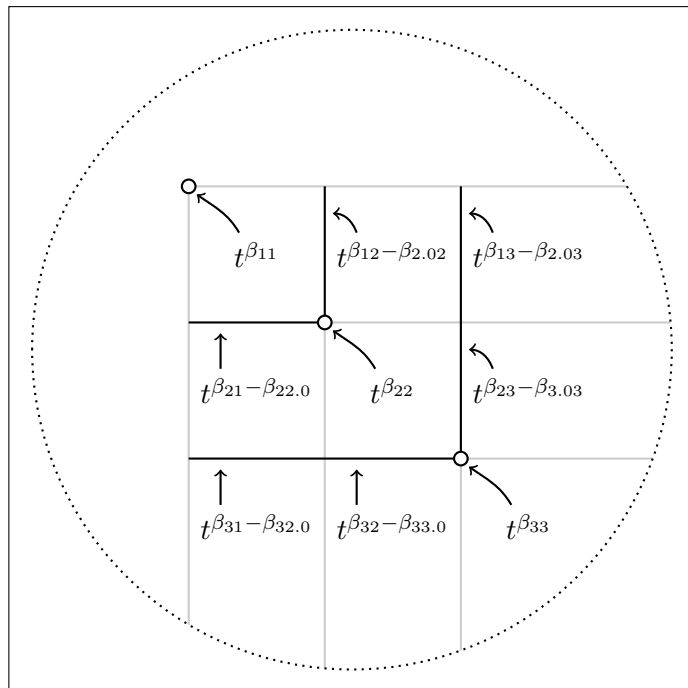
Theorem 6.3. *Let β be a plane partition with associated weight exponents $c(\beta)$, and let w be an essential weighting of Γ_0 with weight exponents $c(w)$. If $c(\beta) = c(w)$, then the arc $\Lambda(\Gamma_0, w)$ belongs to the contact stratum \mathcal{C}_β .*

The simplest arc provided by Theorem 6.3 is obtained by setting $w_{ij} = t^{c_{i,j}}$. For example, the weighted network $(\Gamma_0, t^{c_{i,j}})$ for $G(3,6)$ is given in Fig. 6E, and the matrix $X(\Gamma_0, t^{c_{i,j}}) = (x_{i,j})$ is given in Fig. 6F. Notice how the entries of the resulting matrix have many terms. In some cases, it is possible to exhibit simpler arcs in a contact stratum, in the sense that their matrices are more sparse, have more zeros. For example, for the plane partition

$$\beta = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

our method constructs the arc

$$\begin{pmatrix} t^2 + t^3 & t^2 & 0 & 1 \\ t^2 & t & 1 & 0 \end{pmatrix},$$

FIGURE 6E. The weighted network $(\Gamma_0, t^{c_{i,j}})$ for $G(3,6)$.

$x_{11} = t^{\beta_{11}}$ $+ t^{\beta_{12}+\beta_{21}-\beta_{22}} + t^{\beta_{12}+\beta_{31}-\beta_{32}}$ $+ t^{\beta_{13}+\beta_{21}-\beta_{23}} + t^{\beta_{13}+\beta_{31}-\beta_{33}}$ $+ t^{\beta_{13}+\beta_{22}+\beta_{31}-\beta_{23}-\beta_{32}}$	$x_{12} = t^{\beta_{12}}$ $+ t^{\beta_{13}+\beta_{22}-\beta_{23}}$ $+ t^{\beta_{13}+\beta_{32}-\beta_{33}}$	$x_{13} = t^{\beta_{13}}$
$x_{21} = t^{\beta_{21}}$ $+ t^{\beta_{22}+\beta_{31}-\beta_{32}}$ $+ t^{\beta_{23}+\beta_{31}-\beta_{33}}$	$x_{22} = t^{\beta_{22}}$ $+ t^{\beta_{23}+\beta_{32}-\beta_{33}}$	$x_{23} = t^{\beta_{23}}$
$x_{31} = t^{\beta_{31}}$	$x_{32} = t^{\beta_{32}}$	$x_{33} = t^{\beta_{33}}$

FIGURE 6F. The matrix $X(\Gamma_0, t^{c_{i,j}})$ in $G(3,6)$ for a generic plane partition $\beta = (\beta_{i,j})$. It gives an arc in the contact stratum \mathcal{C}_β .

but the following arc also belongs to \mathcal{C}_β :

$$\begin{pmatrix} t^2 & 0 & 0 & 1 \\ 0 & t & 1 & 0 \end{pmatrix}.$$

In general, it seems hard to give an algorithm producing sparse examples.

We proceed now to prove Theorem 6.3. For the rest of the section we assume the hypotheses of the theorem: we fix a plane partition β and an essential weighting w with $c(\beta) = c(w)$. We start with the following easy statement, which in fact motivated our definition of $c(\beta)$.

Lemma 6.4. *Let $M_{i,j}$ be a final minor of $X(\Gamma_0, w)$, and set $r = \min\{k - i, n - k - j\}$. Then:*

$$\text{ord}_t(M_{i,j}) = \beta_{i,j} + \beta_{i+1,j+1} + \cdots + \beta_{i+r,j+r}.$$

Proof. This is an immediate consequence of Lemma 6.2. \square

Let $M = [a_0 \dots a_r | b_0 \dots b_r]$ be a minor of $X(\Gamma_0, w)$ of size $r + 1$. We define

$$M' = M_{k-r, b_r-r} = [k - r \dots k | b_r - r \dots b_r]$$

and

$$M'' = M_{a_r-r, n-k-r} = [a_r - r \dots a_r | n - k - r \dots n - k].$$

Notice that both M' and M'' are final minors, and that $M \leq M'$ and $M \leq M''$ (according to the order on minors introduced in Section 2).

Lemma 6.5. *Let M , M' , and M'' be as above. Then:*

$$\text{ord}_t(M) \geq \text{ord}_t(M') \quad \text{and} \quad \text{ord}_t(M) \geq \text{ord}_t(M'').$$

Proof. We focus on the first inequality, the proof of the second one is analogous. Using Lemma 6.1 we get an expansion

$$M = \sum w_{p_0} \cdots w_{p_r},$$

the sum ranging among all non-intersecting collections of paths $\{p_0, \dots, p_r\}$ where p_s connects the source a_s with the sink b_s . From Lemma 6.2 we know that

$$M' = w_{q_0} \cdots w_{q_r},$$

where q_s is the path that starts at the source $k - r + s$, then moves horizontally to the left until the column $b_r - s$, and then moves vertically down until the sink $b_r - s$. Observe that $\text{ord}_t(w_{q_s}) = \beta_{k-r+s, b_r-s}$. The lemma will follow if we show that $\text{ord}_t(w_{p_s}) \geq \text{ord}_t(w_{q_s})$ for all the possible collections of paths $\{p_s\}$ and all s .

Fix one such collection $\{p_s\}$, and $0 \leq s \leq r$. In Γ_0 , the vertices that have weights are disposed along a diagonal, in such a way that any maximal path must pass through one of them. Let v^0 be the (only) weighted vertex in the path p_s , and let \tilde{p}_s be the final part of the path p_s that connects v^0 with the sink b_s . Let NW be the collection of north-west corners of \tilde{p}_s ; these are the vertices v of \tilde{p}_s for which no other vertex of \tilde{p}_s is immediately above or immediately to the left of v . We write $NW = \{v_{i_1, j_1}, \dots, v_{i_\ell, j_\ell}\}$, where the vertices are ordered using the orientation of \tilde{p}_s , and $v_{i,j}$ denotes the vertex of Γ_0 in the grid position (i, j) . Observe that it could happen that $v^0 = v_{i_1, j_1}$. From the definitions it follows that

$$\text{ord}_t(w_{\tilde{p}_s}) = \beta_{i_1, j_1} + \beta_{i_2, j_2} - \beta_{i_2, j_1} + \cdots + \beta_{i_\ell, j_\ell} - \beta_{i_\ell, j_{(\ell-1)}},$$

and, using the fact that β is a plane partition,

$$\text{ord}_t(w_{\tilde{p}_s}) \geq \beta_{i_\ell, j_\ell}.$$

By construction, the path p_s must be above the path q_s , and therefore the lowest north-west corner of p_s (with is v_{i_ℓ, j_ℓ}) must be to the north-west of the (only) north-west corner of q_s (which is v_{k-r+s, b_r-s}). In particular

$$i_\ell \leq k - r + s \quad \text{and} \quad j_\ell \leq b_r - s.$$

Using again that β is a plane partition we see that

$$\text{ord}_t(w_{p_s}) \geq \text{ord}_t(w_{\tilde{p}_s}) \geq \beta_{i_\ell, j_\ell} \geq \beta_{k-r+s, b_r-s} = \text{ord}_t(w_{q_s}),$$

as required. \square

Proof of Theorem 6.3. The theorem follows if we show that

$$\text{ord}_\Lambda(\Omega_{j^i}) = \text{ord}_t(M_{i,j}),$$

where $\Lambda = \Lambda(\Gamma_0, w)$ and $M_{i,j}$ is a final minor of $X(\Gamma_0, w)$. From Proposition 2.4 it is enough to show that

$$\text{ord}_t(M) \geq \text{ord}_t(M_{i,j}), \quad (6a)$$

for all minors M such that $M \leq M_{i,j}$. Consider M' and M'' as Lemma 6.5. Then, when $(k-i) \leq (n-k-j)$ we have that $M \leq M' \leq M_{i,j}$, and when $(k-i) \geq (n-k-j)$ we have $M \leq M'' \leq M_{i,j}$. Therefore, using Lemma 6.5, we see that it is enough to prove Eq. (6a) in the case where M is a final minor. But this case is a consequence of Lemma 6.4 and the fact that β is a plane partition. \square

7. Schubert valuations

The main goal of this section is to prove the following statement.

Theorem 7.1. *Every contact stratum \mathcal{C}_β is an irreducible subset of $J_\infty G(k, n)$.*

Definition 7.2. From Theorem 7.1 it follows that the closure $\overline{\mathcal{C}}_\beta$ is the maximal arc set in $J_\infty G(k, n)$ associated to the semi-valuation ord_β . These semi-valuations are called *Schubert semi-valuations*.

Observe that \mathcal{C}_β is a contact locus precisely when β is a plane partition with finite height (no infinities allowed). In this case ord_β is a valuation (and not just a semi-valuation). As we will see, Schubert valuations are the most relevant from the point of view of the study of the singularities of Schubert varieties.

For a Schubert semi-valuation ord_β , we can easily determine its home and its center from the plane partition β . We let β^1 be the base of β : the linear partition whose diagram contains the positions (i, j) where $\beta_{i,j} \geq 1$. Analogously, β^∞ is the partition corresponding to the condition $\beta_{i,j} = \infty$. Then the home of ord_β is the Schubert variety Ω_{β^∞} , and its center is Ω_{β^1} .

To prove Theorem 7.1 we will use the techniques developed in Section 6 and produce explicitly the generic point of \mathcal{C}_β . We consider the torus $(\mathbb{C}^\times)^{k(n-k)}$, and its arc space $J_\infty(\mathbb{C}^\times)^{k(n-k)} = (\mathbb{C}[[t]]^\times)^{k(n-k)}$. Notice that this arc space is a connected algebraic group, and in particular it is irreducible. Its generic point is a matrix that we denote $(u_{i,j})$. Its entries are of the form

$$u_{i,j} = u_{i,j}^{[0]} + u_{i,j}^{[1]}t + u_{i,j}^{[2]}t^2 + \cdots + u_{i,j}^{[p]}t^p + \cdots$$

where the coefficients $u_{i,j}^{[p]}$ are transcendentals generating the function field of the arc space of the torus:

$$\mathbb{C}\left(J_\infty(\mathbb{C}^\times)^{k(n-k)}\right) = \mathbb{C}\left(u_{i,j}^{[p]} \mid 1 \leq i \leq k, \quad 1 \leq j \leq n-k, \quad 0 \leq p \leq \infty\right).$$

Fix a plane partition β , possibly with infinite height. With the notations of Section 6, we define an essential weighting $w = w(\beta, u)$ on Γ_0 given by

$$w_{i,j} = t^{c_{i,j}} u_{i,j},$$

where the $c_{i,j}$ are the weight exponents associated to β . Recall that we use the notations

$$X(\Gamma_0, w(u, \beta)) \quad \text{and} \quad \Lambda(\Gamma_0, w(u, \beta))$$

for the weight matrix and arc associated to this weighting. We think of them as giving a morphism between arc spaces:

$$\Phi_\beta: J_\infty(\mathbb{C}^\times)^{k(n-k)} \rightarrow J_\infty G(k, n), \quad u \mapsto \Lambda(\Gamma_0, w(u, \beta)).$$

Notice that it follows from Theorem 6.3 that the image of Φ_β is contained in \mathcal{C}_β . Also, from Lemma 6.2 it is easy to see that the minors $M_{i,j}$ determine the weights $w_{i,j}$, and therefore the morphism Φ_β is injective.

Lemma 7.3. *Let Λ be an arc in the contact stratum \mathcal{C}_β .*

- (1) *Λ belongs to the image of Φ_β if and only if it is contained in the opposite big cell and $\text{ord}_\Lambda(\Omega_{(j^i)}) = \text{ord}_\Lambda(M_{i,j})$ for each final minor $M_{i,j}$.*
- (2) *The Borel orbit $\Lambda \cdot B$ has non-empty intersection with the image of Φ_β .*

Proof of Theorem 7.1. Consider the morphism:

$$\Psi_\beta: B \times J_\infty(\mathbb{C}^\times)^{k(n-k)} \rightarrow J_\infty G(k, n), \quad (b, u) \mapsto \Lambda(\Gamma_0, w(u, \beta)) \cdot b.$$

From Lemma 7.3, part 2, we see that the image of Ψ_β is the whole contact stratum \mathcal{C}_β . Since the domain of Ψ_β is irreducible, the theorem follows. \square

Proof of Lemma 7.3, part 2. Let Λ' be a generic B -translate of Λ . Since all Schubert varieties intersect the opposite big cell, we know that Λ' is contained in the opposite big cell. Let $M \leq M_{i,j}$ be the minor of Λ' for which $\text{ord}_{\Lambda'}(\Omega_{(j^i)}) = \text{ord}_{\Lambda'}(M)$. The action of B on Λ' transforms $M_{i,j}$ into a linear combination of minors \tilde{M} of the same size verifying $\tilde{M} \leq M_{i,j}$. Moreover, if the action is by a generic element of B , all such minors appear in the linear combination. In particular, since Λ' is already a generic translate, we see that $\text{ord}_{\Lambda'}(M) = \text{ord}_{\Lambda'}(M_{i,j})$. Now the result follows from part 1. \square

Proof of Lemma 7.3, part 1. The necessary condition is an immediate consequence of Lemma 6.5 and Lemma 6.4. In fact this was already shown during the proof of Theorem 6.3, as Eq. (6a).

For the sufficient condition, let Λ be an arc satisfying the hypothesis, and consider the final minors $M_{i,j}$ of Λ . It follows from Lemma 6.2 that we can find a u for which $\Lambda(\Gamma_0, w(u, \beta))$ also has the $M_{i,j}$ as its final minors. The result follows if we show that $\Lambda = \Lambda(\Gamma_0, w(u, \beta))$. For this we use the argument of [FZ00, Lemma 7], adapted to allow for the possibility of some final minors being zero.

Let $X = (x_{i,j})$ and $X(\Gamma_0, w(u, \beta)) = (\tilde{x}_{i,j})$ be the matrices determining Λ and $\Lambda(\Gamma_0, w(u, \beta))$ in the opposite big cell. We know that the final minors of these two matrices agree. Also, notice that both $(x_{i,j})$ and $(\tilde{x}_{i,j})$ verify the hypothesis, which we rewrite as

$$\begin{aligned} \text{ord}_t(M_{i,j}) &= \min \{ \text{ord}_t(M) \mid M \text{ a minor of } (x_{i,j}), M \leq M_{i,j} \} \\ &= \min \{ \text{ord}_t(\tilde{M}) \mid \tilde{M} \text{ a minor of } (\tilde{x}_{i,j}), \tilde{M} \leq M_{i,j} \}. \end{aligned} \tag{7a}$$

We prove by induction that $x_{i,j} = \tilde{x}_{i,j}$. The base case is when $i = k$ or $j = n - k$, which clearly implies $M_{i,j} = x_{i,j} = \tilde{x}_{i,j}$. Otherwise we have an expansion

$$M_{i,j} = x_{i,j} M_{i+1,j+1} + P(x)$$

where $P(x)$ is a polynomial in entries $x_{i',j'}$ where $i' \geq i$, $j' \geq j$, and $(i', j') \neq (i, j)$. Analogously,

$$M_{i,j} = \tilde{x}_{i,j} M_{i+1,j+1} + P(\tilde{x})$$

where $P(\tilde{x})$ is obtained from $P(x)$ by replacing each $x_{i',j'}$ with $\tilde{x}_{i',j'}$. By induction we see that

$$x_{i,j}M_{i+1,j+1} = \tilde{x}_{i,j}M_{i+1,j+1}.$$

If $M_{i+1,j+1} \neq 0$ we conclude. Assume $M_{i+1,j+1} = 0$, and consider the minor M' obtained from $M_{i,j}$ by removing the row $i+1$ and the column $j+1$ in X . Notice that $M' \leq M_{i+1,j+1}$, and by Eq. (7a) this implies that $M' = 0$. We construct similarly \tilde{M}' from $X(\Gamma_0, w(u, \beta))$, and we also get $\tilde{M}' = 0$. If M and M' have size 1×1 , we get $x_{i,j} = \tilde{x}_{i,j} = M = 0$ and we conclude. Otherwise, expanding M and M' we get

$$x_{i,j}M_{i+2,j+2} + P'(x) = \tilde{x}_{i,j}M_{i+2,j+2} + P'(\tilde{x}).$$

Again the induction hypothesis implies that $P'(x) = P'(\tilde{x})$. Now we can repeat the same argument: we conclude if $M_{i+2,j+2} \neq 0$, and otherwise we consider $M'' = \tilde{M}'' = 0$ by removing the row $i+2$ and the column $j+2$. Eventually this process must stop, showing that $x_{i,j} = \tilde{x}_{i,j}$. \square

8. The generalized Nash problem for contact strata

In this section we start the analysis of the finer geometric structure of contact strata. We consider the closures $\overline{\mathcal{C}}_\beta$ of contact strata inside of the arc space of the Grassmannian, which we simply call the *closed contact strata*. We are mainly interested in the following version of the *generalized Nash problem*:

Problem 8.1. *Determine all possible containments among closed contact strata.*

As it often happens with Nash-type questions, this problem seems to be very difficult. Nevertheless, we are able to show several types of containments among closed contact strata, and that will be enough for the applications in the rest of the paper.

The Plücker order. We start with the easier direction: a necessary condition for a containment to exist. Given a plane partition β , we consider the weighting $w(u, \beta)$ as in Section 7, so that $\Lambda(\Gamma_0, w(u, \beta))$ is the generic point of the contact stratum \mathcal{C}_β .

Now we consider a Plücker coordinate $[i_1 \dots i_k]$. Then the number

$$\text{ord}_\beta([i_1 \dots i_k]) = \text{ord}_{\Lambda(\Gamma_0, w(u, \beta))}([i_1 \dots i_k])$$

is well-defined, in the sense that it only depends on β and $[i_1 \dots i_k]$. Notice that there are combinatorial descriptions of Γ_0 and $w(u, \beta)$, so one could implement an algorithm to compute these orders. We use these numbers to define an order among plane partitions.

Definition 8.2. We say that β is less than or equal to β' in the *Plücker order*, written $\beta \leq \beta'$, if

$$\text{ord}_\beta([i_1 \dots i_k]) \leq \text{ord}_{\beta'}([i_1 \dots i_k])$$

for all Plücker coordinates $[i_1 \dots i_k]$.

The following is obvious from the definitions.

Lemma 8.3. *Consider two plane partitions β and β' . Then:*

$$\overline{\mathcal{C}}_\beta \supseteq \overline{\mathcal{C}}_{\beta'} \quad \Rightarrow \quad \beta \leq \beta'.$$

In particular, if α (resp. α') is the contact profile of any arc in \mathcal{C}_β (resp. of any arc in $\mathcal{C}_{\beta'}$), then we have that

$$\overline{\mathcal{C}}_\beta \supseteq \overline{\mathcal{C}}_{\beta'} \quad \Rightarrow \quad \alpha \leq \alpha'.$$

Here $\alpha \leq \alpha'$ means that $\alpha_\lambda \leq \alpha'_\lambda$ for all partitions λ .

The condition in Lemma 8.3 is not sufficient in general to guarantee a containment of closed contact strata. An example of this is given in $G(3, 6)$ by the following plane partitions:

$$\beta = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \beta' = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

It is possible to check (although quite tedious) that $\beta \leq \beta'$. But the partitions have the same number of boxes, and we will see in Proposition 9.3 that this prevents the existence of a containment.

On the other hand, in the special case of $G(2, 4)$ the Plücker order completely characterizes containments.

Proposition 8.4. *Consider two contact strata \mathcal{C}_β and $\mathcal{C}_{\beta'}$ in $J_\infty G(2, 4)$. Then:*

$$\overline{\mathcal{C}}_\beta \supseteq \overline{\mathcal{C}}_{\beta'} \Leftrightarrow \beta \leq \beta'.$$

Proof. Using the notation of Section 7, we consider the matrix $X = X(\Gamma_0, w(u, \beta))$. A direct computation gives the following:

$$X = \begin{pmatrix} u_{11}t^{\beta_{11}} + u_{12}u_{21}u_{22}t^{\beta_{12}+\beta_{21}-\beta_{22}} & u_{12}u_{22}t^{\beta_{12}} \\ u_{21}u_{22}t^{\beta_{21}} & u_{22}t^{\beta_{22}} \end{pmatrix}.$$

Therefore, the orders of contact with respect to the Plücker coordinates are:

$[ij]$	$[12]$	$[13]$	$[14]$	$[23]$	$[24]$	$[34]$
$\text{ord}_\beta([ij])$	$\beta_{11} + \beta_{22}$	$\min\{\beta_{11}, \beta_{12} + \beta_{21} - \beta_{22}\}$	β_{21}	β_{12}	β_{22}	0

and the Plücker order is given by:

$$\beta \leq \beta' \Leftrightarrow \begin{cases} \min\{\beta_{11}, \beta_{12} + \beta_{21} - \beta_{22}\} \leq \min\{\beta'_{11}, \beta'_{12} + \beta'_{21} - \beta'_{22}\}, \\ \beta_{11} + \beta_{22} \leq \beta'_{11} + \beta'_{22}, \quad \beta_{21} \leq \beta'_{21}, \quad \beta_{12} \leq \beta'_{12}, \quad \beta_{22} \leq \beta'_{22}. \end{cases}$$

From this explicit description, it is easy to determine the covers for the Plücker order. In precise terms, if $\beta \triangleleft \beta'$, then we can find a partition β^* verifying $\beta \triangleleft \beta^* \leq \beta'$, and such that the difference $\beta^* - \beta$ is one of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

It is therefore enough to prove the proposition assuming $\beta' - \beta$ is one of the six matrices above. The first four cases are easy, and the last two are transposed of each other, so we will focus on the last case:

$$\begin{aligned} \beta'_{11} &= \beta_{11} - 1, & \beta'_{12} &= \beta_{12}, \\ \beta'_{21} &= \beta_{21} + 1, & \beta'_{22} &= \beta_{22} + 1. \end{aligned}$$

Observe that since $\beta \triangleleft \beta'$, we must have that $\beta_{11} > \beta_{12} + \beta_{21} - \beta_{22}$.

Consider the matrix $X' = X(\Gamma_0, w(u', \beta'))$ giving the generic point of $\mathcal{C}_{\beta'}$:

$$X' = \begin{pmatrix} x'_{11} & x'_{12} \\ x'_{21} & x'_{22} \end{pmatrix} = \begin{pmatrix} u'_{11}t^{\beta_{11}-1} + u'_{12}u'_{21}u'_{22}t^{\beta_{12}+\beta_{21}-\beta_{22}} & u'_{12}u'_{22}t^{\beta_{12}} \\ u'_{21}u'_{22}t^{\beta_{21}+1} & u'_{22}t^{\beta_{22}+1} \end{pmatrix},$$

and let $X(s)$ be the one-parameter family of matrices given by

$$X(s) = \begin{pmatrix} x'_{11} & x'_{12} \\ x'_{21} + sx'_{11}t^v & x'_{22} + sx'_{12}t^v \end{pmatrix}$$

where $v = \beta_{22} - \beta_{12}$. Clearly $X(0) = X'$, and direct computation shows that $X(s) \in \mathcal{C}_\beta$ for generic s . For this, the only non-obvious thing to check is that

$$\text{ord}_t(x'_{21} + sx'_{11}t^v) = \beta_{21}. \quad (8a)$$

But we have that

$$\text{ord}_t(x'_{21} + sx'_{11}t^v) = \min\{\beta_{21} + 1, \beta_{11} + \beta_{22} - \beta_{12} - 1, \beta_{21}\},$$

and hence Eq. (8a) follows from the inequality $\beta_{11} > \beta_{12} + \beta_{21} - \beta_{22}$.

We have constructed a family $X(s)$ whose special point is the generic point of $\mathcal{C}_{\beta'}$ and whose generic point is in \mathcal{C}_β . Therefore $\overline{\mathcal{C}}_\beta \supseteq \mathcal{C}_{\beta'}$, as required. \square

Nash containments via weight exponents. The easiest way to give a general sufficient condition for a containment $\overline{\mathcal{C}}_\beta \supseteq \overline{\mathcal{C}}_{\beta'}$ is via an analysis of the weight exponents.

Theorem 8.5. *Let β and β' be plane partitions, and let $c(\beta) = (c_{i,j})$ and $c(\beta') = (c'_{i,j})$ be the corresponding weight exponents (as in Section 6). Assume that $c(\beta) \leq c(\beta')$ (i.e., that $c_{i,j} \leq c'_{i,j}$ for all i, j). Then $\overline{\mathcal{C}}_\beta \supseteq \overline{\mathcal{C}}_{\beta'}$.*

Proof. In the notation of Section 7, we write the generic points of \mathcal{C}_β and $\mathcal{C}_{\beta'}$ as

$$\Lambda(\Gamma_0, w) \quad \text{and} \quad \Lambda(\Gamma_0, w')$$

where $w = w(u, \beta)$ and $w' = w(u', \beta')$ are the essential weightings of Γ_0 given by

$$w_{i,j} = t^{c_{i,j}} u_{i,j} \quad \text{and} \quad w'_{i,j} = t^{c'_{i,j}} u'_{i,j}.$$

Also, recall that each $u_{i,j}$ is a power series of the form

$$u_{i,j} = u_{i,j}^{[0]} + u_{i,j}^{[1]}t + u_{i,j}^{[2]}t^2 + \cdots + u_{i,j}^{[p]}t^p + \cdots$$

where the $u_{i,j}^{[p]}$ are variables. We have an analogous description of $u'_{i,j}$.

The theorem will follow if we write an specialization of $\Lambda(\Gamma_0, w)$ to $\Lambda(\Gamma_0, w')$. To do this, it is enough to describe how each variable $u_{i,j}^{[p]}$ specializes to a function of the $u'_{i,j}^{[p]}$. We consider the specialization given by

$$u_{i,j}^{[p]} \mapsto \begin{cases} 0 & \text{if } p < c'_{i,j} - c_{i,j}, \\ u'_{i,j}^{[p-q]} & \text{if } p \geq q = c'_{i,j} - c_{i,j}. \end{cases}$$

Observe that under this specialization we have

$$w_{i,j} = t^{c_{i,j}} u_{i,j} \mapsto t^{c'_{i,j}} u'_{i,j} = w'_{i,j},$$

and therefore $\Lambda(\Gamma_0, w)$ specializes to $\Lambda(\Gamma_0, w')$, as required. \square

Nash containments via plateaux. Given plane partitions $\beta \triangleleft \beta'$, we are going to give some subtle conditions on the shapes of β and β' that guarantee the existence of a containment $\overline{\mathcal{C}}_\beta \supsetneq \overline{\mathcal{C}}_{\beta'}$. For this we need some definitions.

Definition 8.6. We say that a plane partition β has a *plateau* up to position (a, b) if there is a number h such that $\beta_{i,j} = h$ when $i \leq a, j \leq b$, and $(i, j) \neq (a, b)$. The position (a, b) is called the *corner* of the plateau, and h is called the *height* of the plateau. If $(a, b) = (1, 1)$ we set $h = \infty$. If the height is finite, the difference $h - \beta_{a,b}$ is called the *fall*. If the height is infinite, we say that the fall is 0 if $\beta_{a,b} = \infty$, and that it is ∞ if $\beta_{a,b} < \infty$.

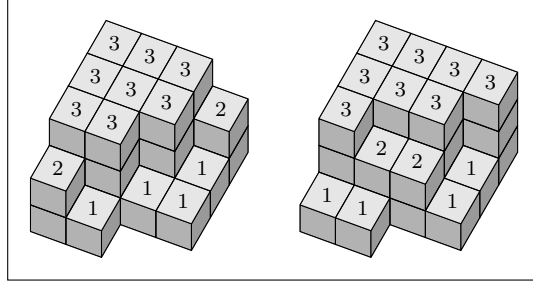


FIGURE 8A. Some examples of plateaux in $G(4,8)$. The plane partition on the left has three plateaux with positive finite fall: their corners are in positions $(1,4)$, $(3,3)$, and $(4,1)$. On the right, $(2,4)$, $(3,2)$, and $(4,1)$ are the corners of the plateaux with positive finite fall. All of these plateaux have height 3.

Notice that the above definition does not impose any condition on the entry in the corner (the one in position (a,b)). Also, all plane partitions with finite height have a plateau with infinite fall and corner at $(1,1)$. For some examples see Fig. 8A.

Let β be a plane partition with a plateau with corner at (a,b) . Let h and f be the height and fall of the plateau. We want to understand the orders of the weights in the network $(\Gamma_0, w(u, \beta))$. These are determined by the weight exponents $c(\beta) = (c_{i,j})$. From the definitions we see that the plateau imposes some conditions on $c(\beta)$. More precisely, if (i,j) verifies

$$i \leq a, \quad j \leq b, \quad \text{and} \quad (i,j) \notin \{(a,b), (a-1,b), (a,b-1)\},$$

then we have

$$\begin{aligned} c_{i,j} &= 0 & \text{if } (k-i) \neq (n-k-j), \text{ and} \\ c_{i,j} &= h & \text{if } (k-i) = (n-k-j). \end{aligned}$$

We also have some information on the values of $c_{a-1,b}$ and $c_{a,b-1}$. We have three possibilities:

$$\begin{aligned} (k-a) &= (n-k-b) & \Rightarrow & c_{a-1,b} = c_{a,b-1} = f, \\ (k-a) &< (n-k-b) & \Rightarrow & c_{a-1,b} = f, \\ (k-a) &> (n-k-b) & \Rightarrow & c_{a,b-1} = f. \end{aligned} \tag{8b}$$

In the cases $a = 1$ or $b = 1$, the above equations that involve $c_{a-1,b}$ and $c_{a,b-1}$ are to be ignored.

We want to explain the consequences of having the plateau in terms of the weighted planar network. We denote by $NW_{a,b}$ the part of the grid in Γ_0 to the north-west of the vertex in position (a,b) . We let $L_{a,b}$ be the open south-east corner of $NW_{a,b}$, containing the vertex in position (a,b) , the vertical edge above (a,b) , and the horizontal edge to the left of (a,b) . $L_{a,b}$ only contains the interior of the two edges, so there is only one vertex in $L_{a,b}$. We set $NW_{a,b}^\circ = NW_{a,b} \setminus L_{a,b}$.

The conditions found above on the weight exponents $c_{i,j}$ can be translated in terms of the network as follows. Inside $NW_{a,b}^\circ$ all the edges have weights of order 0, and all the vertices have weights of order 0 or h . The edges of $L_{a,b}$ have weights of order either 0 or f , corresponding to the different cases of Eq. (8b). See Fig. 8B.

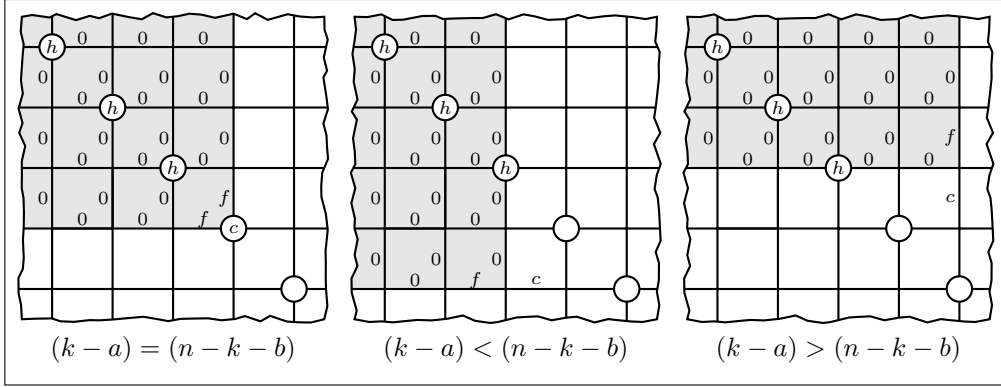


FIGURE 8B. The consequences of having a plateau in β . The shaded region corresponds to $NW_{a,b}$. The edges and weighted vertices are labeled with the order of the weight. h and f are the height and fall of the plateau, and $c = c_{a,b}$.

Theorem 8.7. *Let β be a plane partition having a plateau with corner at (a, b) . Assume the plateau has positive fall, and let β' be the plane partition obtained by adding one box in position (a, b) . Then there is a containment $\overline{\mathcal{C}}_\beta \supsetneq \overline{\mathcal{C}}_{\beta'}$.*

Proof. We will produce a wedge realizing the containment. That is, we will construct a one-parameter family of arcs Λ_s such that the generic point of the family belongs to \mathcal{C}_β and the special point of the family is the generic point of $\mathcal{C}_{\beta'}$.

Let s be a transcendental, meant to be the parameter of the family we are about to construct.

We let Γ_1 be the planar network obtained from Γ_0 by adding one diagonal edge joining $v_{a,b+1}$ with $v_{a+1,b}$. Here $v_{i,j}$ denotes the internal vertex of Γ_0 in position (i, j) , $v_{i,n-k+1}$ is the source labeled i , and $v_{k+1,j}$ is the sink labeled j . We construct a weighting $w(s)$ of Γ_1 in the following way. The new diagonal edge of Γ_1 gets weight $st^{c_{a,b}}$, where $c(\beta) = (c_{i,j})$ are the weight exponents of β . The rest of edges of Γ_1 , and all the vertices, get the same weight as in $(\Gamma_0, w(u, \beta'))$. See Fig. 8C for the local structure of $(\Gamma_1, w(s))$ around the added diagonal edge.

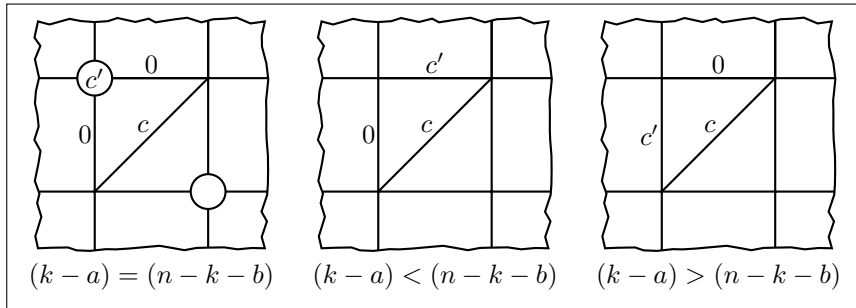


FIGURE 8C. The local structure of $(\Gamma_1, w(s))$. Edges and weighted vertices are labeled with the order of the weight. $c = c_{a,b}$ and $c' = c'_{a,b}$ are weight exponents for β and β' .

We have a morphism

$$\mathbb{A}^1 \longrightarrow J_\infty G(k, n), \quad s \mapsto \Lambda(\Gamma_1, w(s)).$$

When $s = 0$, the added diagonal edge in Γ_1 gets weight 0, and therefore it does not affect the weight matrix. Hence $\Lambda(\Gamma_1, w(0)) = \Lambda(\Gamma_0, w(u, \beta'))$. The theorem follows if we show that $\Lambda(\Gamma_1, w(s))$ belongs to \mathcal{C}_β for generic s .

To lighten notation, we will simply denote by w_p the weight of a path p in the weighted network $(\Gamma_0, w(u, \beta))$, and by w'_q the weight of a path in $(\Gamma_1, w(s))$. Notice that if the path p is contained in Γ_0 , then w'_p coincides with the weight of p with respect to $(\Gamma_0, w(u, \beta'))$.

To determine the invariant factor profile of $\Lambda(\Gamma_1, w(s))$ we first compute the orders of its final minors. Let $M = M_{i_0, j_0}$ be a final minor of $\Lambda(\Gamma_1, w(s))$ with row set $I = [i_0 \dots i_r]$ and column set $J = [j_0 \dots j_r]$. Our goal is to show the following:

$$\text{ord}_t(M) = \beta_{i_0, j_0} + \dots + \beta_{i_r, j_r}. \quad (8c)$$

We know that there is a unique collection of paths $P = \{p_0, \dots, p_r\}$ in Γ_0 connecting I with J . These paths verify:

$$\text{ord}_t(w'_{p_0}) = \beta'_{i_0, j_0}, \quad \text{ord}_t(w'_{p_1}) = \beta'_{i_1, j_1}, \quad \dots \quad \text{ord}_t(w'_{p_r}) = \beta'_{i_r, j_r}.$$

Also, recall that:

$$\begin{aligned} \beta'_{i, j} &= \beta_{i, j} & \text{if } (i, j) \neq (a, b), \text{ and} \\ \beta'_{a, b} &= \beta_{a, b} + 1. \end{aligned}$$

If none of the pairs (i_0, j_0) equals (a, b) , then P is also the unique collection of paths in Γ_1 connecting I with J . In this case it is easy to show that Eq. (8c) holds:

$$\text{ord}_t(M) = \text{ord}_t(w'_P) = \beta'_{i_0, j_0} + \dots + \beta'_{i_r, j_r} = \beta_{i_0, j_0} + \dots + \beta_{i_r, j_r}.$$

Assume that $(i_\ell, j_\ell) = (a, b)$ for some $0 \leq \ell \leq r$. Write $M = \sum w'_Q$, where $Q = \{q_0, \dots, q_r\}$ ranges among the collections of paths in Γ_1 connecting I with J . For each such Q the last paths agree with the ones of P , more precisely:

$$p_{\ell+1} = q_{\ell+1}, \quad p_{\ell+2} = q_{\ell+2}, \quad \dots \quad p_r = q_r. \quad (8d)$$

There are two possibilities for q_ℓ . It could be $q_\ell = p_\ell$, in which case we also have $P = Q$. Otherwise q_ℓ is the path \hat{p}_ℓ obtained from p_ℓ by removing the north-west corner and adding the diagonal edge of Γ_1 . From the computation in Fig. 8C we see that the weight of \hat{p}_ℓ is given by:

$$\text{ord}_t(w'_{\hat{p}_\ell}) = \text{ord}_t(w'_{p_\ell}) - c'_{a, b} + c_{a, b} = \beta'_{a, b} - c'_{a, b} + c_{a, b} = \beta_{a, b}.$$

We consider the collection \hat{P} obtained from P by replacing p_ℓ with \hat{p}_ℓ . Observe that:

$$\text{ord}_t(w'_{\hat{P}}) = \text{ord}_t(w'_P) - \beta'_{a, b} + \beta_{a, b} = \beta_{i_0, j_0} + \dots + \beta_{i_r, j_r}.$$

Therefore, in order to finish the proof of Eq. (8c), it is enough to show that

$$\text{ord}_t(w'_Q) \geq \text{ord}_t(w'_{\hat{P}}). \quad (8e)$$

If $q_\ell = p_\ell$, then $Q = P$ and Eq. (8e) follows from the fact that $\beta'_{a, b} = \beta_{a, b} + 1$. We can therefore assume that $q_\ell = \hat{p}_\ell$.

The paths $q_0, \dots, q_{\ell-1}$ are contained in Γ_0 , and agree with the paths $p_0, \dots, p_{\ell-1}$ away from the region $NW_{a, b}$. From the discussion summarized in Fig. 8B, we see that we have

$$\text{ord}_t(w'_{q_0}) = \text{ord}_t(w'_{p_0}), \quad \dots \quad \text{ord}_t(w'_{q_{\ell-2}}) = \text{ord}_t(w'_{p_{\ell-2}}). \quad (8f)$$

For $q_{\ell-1}$ there are two possibilities. We can have $q_{\ell-1} = p_{\ell-1}$, which implies $Q = \hat{P}$. Otherwise $q_{\ell-1}$ is obtained from $p_{\ell-1}$ by removing the north-west corner, and replacing it with the corresponding south-west corner. Again using Fig. 8B, we see that

$$\text{ord}_t(w'_{q_{\ell-1}}) = \text{ord}_t(w'_{p_{\ell-1}}) + f'$$

if $(k-a) \neq (n-k-b)$, and

$$\text{ord}_t(w'_{q_{\ell-1}}) = \text{ord}_t(w'_{p_{\ell-1}}) - h' + c'_{a,b} + 2f'$$

if $(k-a) = (n-k-b)$. Here $h' \geq 0$ and $f' \geq 0$ are the height and fall of the plateau in the plane partition β' . Observe that when $(k-a) = (n-k-b)$ we have that $h' = \beta'_{a-1,b-1}$, $c'_{a,b} = \beta'_{a,b}$, and $f' = \beta'_{a-1,j-1} - \beta'_{a,b}$. In all instances, we see that

$$\text{ord}_t(w'_{q_{\ell-1}}) \geq \text{ord}_t(w'_{p_{\ell-1}}). \quad (8g)$$

Combining Eqs. (8d), (8f) and (8g) we get Eq. (8e), and therefore Eq. (8c) is proven.

Recall that we need to prove that $\Lambda(\Gamma_1, w(s))$ has invariant factor profile β . Using Eq. (8c), this amounts to showing that

$$\text{ord}_{\Lambda(\Gamma_1, w(s))}(\Omega_{(j^i)}) = \text{ord}_t(M_{i,j}),$$

where $M_{i,j}$ is a final minor of $\Lambda(\Gamma_1, w(s))$. Equivalently, we need to show that

$$\text{ord}_t(M) \geq \text{ord}_t(M_{i,j}), \quad (8h)$$

for all minors M of $\Lambda(\Gamma_1, w(s))$ such that $M \leq M_{i,j}$ (in the order of minors defined in Section 2). Notice that Eq. (8h) is implied by a version of Lemma 6.2 for the weighed network $(\Gamma_1, w(s))$. But the networks Γ_0 and Γ_1 are very similar, and the proof of Lemma 6.2 can be adapted easily to give the result that we need. \square

9. Log discrepancies

In this section we compute log discrepancies for Schubert valuations. This will be essential for later sections, where we study log canonical thresholds of pairs involving Schubert varieties. The following definitions will be useful.

Definition 9.1. Let β be a plane partition. The *floor* at level s of β is the plane partition of height 1 containing the boxes of β at height s . More precisely, if we denote by μ^s such floor, we have

$$\mu_{i,j}^s = 1 \quad \Leftrightarrow \quad \beta_{i,j} \geq s.$$

Notice that a plane partition of height 1 is determined by its base, so we can also think of the μ^s as linear partitions. A plane partition is determined by its floors. In fact, given a nested sequence of plane partitions of height 1 (or a nested sequence of linear partitions)

$$\mu^1 \supseteq \mu^2 \supseteq \mu^3 \supseteq \cdots \supseteq \mu^h$$

there is a unique plane partition β having μ^s as the floor at level s . Notice that

$$\beta = \mu^1 + \mu^2 + \mu^3 + \cdots + \mu^h$$

Geometrically, we think of β as obtained by stacking the floors on top of each other, with μ^1 on the bottom and μ^h on top.

Definition 9.2. Let β be a plane partition. The *pillar* of β in position (i, j) is the collection of boxes which lay above the position (i, j) in the plane. Notice that the number of boxes of the pillar in position (i, j) is $\beta_{i,j}$, so a plane partition is determined by its pillars.

Proposition 9.3. *Let $\beta = (\beta_{i,j})$ be a plane partition, possibly with infinite height. Then the codimension of \mathcal{C}_β in $J_\infty G(k, n)$ is the number of boxes in β :*

$$\text{codim}(\mathcal{C}_\beta, J_\infty G(k, n)) = \sum_{i,j} \beta_{i,j}$$

Proof. If β has infinite height, then $\mathcal{C}_\beta \subset J_\infty \Omega_\lambda$, where λ is the (linear) partition marking the infinite pillars of β (i.e., the diagram of λ contains the box in position (i, j) if and only if $\beta_{i,j} = \infty$). Since $J_\infty \Omega_\lambda$ has infinite codimension in $J_\infty G(k, n)$, the proposition follows in this case.

Assume that β has finite height, let $h = \beta_{1,1}$ be this height, and let $c = \sum \beta_{i,j}$ be the number of boxes in β . We let $N = hk(n-k)$. We denote by β^0 the empty plane partition, and by β^N the constant plane partition with height h (i.e., $\beta_{i,j}^0 = 0$ and $\beta_{i,j}^N = h$ for all i, j). We will give an algorithm to construct a sequence of nested plane partitions

$$\beta^0 \subset \beta^1 \subset \dots \subset \beta^{c-1} \subset \beta^c \subset \beta^{c+1} \subset \dots \subset \beta^N,$$

such that $\beta^c = \beta$, and such that there are containments $\overline{\mathcal{C}}_{\beta^r} \supsetneq \overline{\mathcal{C}}_{\beta^{r+1}}$ for all $0 \leq r < N$.

The sequence $\beta^0 \subset \dots \subset \beta^c$ corresponds to a process of building the plane partition β by adding one box at a time, and in such a way that the boxes in lower floors are added before the boxes in higher floors. To make this explicit, we order the pillars of a plane partition lexicographically according to their positions. Assuming β^r has been constructed, we consider the boxes of β which are not contained in β^r . Among these, we select the one box which is in the lowest possible floor and in the lexicographically smallest pillar. We add this box to β^r to get β^{r+1} . Notice that in this way the pairs $\beta^r \subset \beta^{r+1}$ satisfy the hypotheses of Theorem 8.7: β^{r+1} is obtained from β^r by adding one box in the corner of a plateau with fall 1. In particular we get containments $\overline{\mathcal{C}}_{\beta^r} \supsetneq \overline{\mathcal{C}}_{\beta^{r+1}}$.

The sequence $\beta^c \subset \dots \subset \beta^N$ corresponds to a process of building the plane partition β^N starting from β^c by adding one box at a time. This can also be done compatibly with Theorem 8.7. For example, we order the pillars of β lexicographically, and at each step we add one box to the pillar which is lexicographically smallest among those having height less than h . Again, we get containments $\overline{\mathcal{C}}_{\beta^r} \supsetneq \overline{\mathcal{C}}_{\beta^{r+1}}$.

For an example of how this algorithm works, see Fig. 9A.

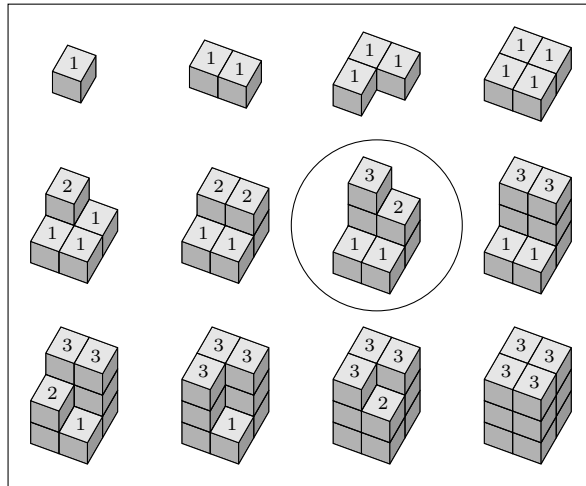


FIGURE 9A. The algorithm of the proof of Proposition 9.3 for the plane partition $\beta = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ in $G(2, 4)$.

We have produced a nested sequence of closed contact strata:

$$\overline{\mathcal{C}}_{\beta^0} \supsetneq \overline{\mathcal{C}}_{\beta^1} \supsetneq \cdots \supsetneq \overline{\mathcal{C}}_{\beta^N}$$

Since closed contact strata are irreducible (by Theorem 7.1), we get lower bounds

$$\text{codim } \mathcal{C}_{\beta^r} \geq r.$$

Let $\mu = ((n-k)^k)$ be the linear partition whose diagram is the rectangle of size $k \times (n-k)$. Then the Schubert variety Ω_μ is just a point (the Borel-fixed point), and it is easy to check that $\text{Cont}^{\geq h}(\Omega_\mu)$ is irreducible and

$$\text{codim } \text{Cont}^{\geq h}(\Omega_\mu) = hk(n-k) = N.$$

Let Λ be the generic point of $\text{Cont}^{\geq h}(\Omega_\mu)$, and let α be the contact profile of Λ . Notice that α is the smallest possible contact profile of all arcs in $\text{Cont}^{\geq h}(\Omega_\mu)$, that is, the smallest contact profile for which $\alpha_\mu \geq h$. This implies that α corresponds to the invariant factor profile β^N , and therefore $\text{Cont}^{\geq h}(\Omega_\mu) \subseteq \overline{\mathcal{C}}_{\beta^N}$. In fact, from the above dimensional considerations, we get the equality $\text{Cont}^{\geq h}(\Omega_\mu) = \overline{\mathcal{C}}_{\beta^N}$. Finally, this implies that $\text{codim } \mathcal{C}_{\beta^r} = r$, and the proposition follows. \square

The following result is an immediate application of Eq. (1a).

Corollary 9.4. *Let $\beta = (\beta_{i,j})$ be a plane partition with finite height. Let ord_β denote the corresponding Schubert valuation, q_β its multiplicity, and $k_\beta = k_{\text{ord}_\beta}(G(k,n))$ its discrepancy. Then*

$$k_\beta + q_\beta = \sum_{i,j} \beta_{i,j}.$$

10. Log canonical thresholds

In this section we study log canonical thresholds of pairs involving Schubert varieties. As mentioned in the introduction, for this we need to introduce a polytope, $\text{SV}(k,n)$, which we call the *polytope of normalized Schubert valuations*. We start by expanding the discussion of the introduction, and describe $\text{SV}(k,n)$ in detail.

The cone of plane partitions. A *plane \mathbb{R} -partition* is a matrix $\beta = (\beta_{i,j})$ of size $k \times (n-k)$, with real coefficients, and verifying the inequalities

$$\beta_{i,j} \geq 0, \quad \beta_{i,j} \geq \beta_{i+1,j}, \quad \text{and} \quad \beta_{i,j} \geq \beta_{i,j+1},$$

whenever they make sense. We denote by $\mathbb{RPP}(k,n) \subset \mathbb{R}^{k(n-k)}$ the set of plane \mathbb{R} -partitions, and we write $\text{PP}(k,n) = \mathbb{RPP}(k,n) \cap \mathbb{Z}^{k(n-k)}$ for its intersection with the standard lattice. Notice that the elements of $\text{PP}(k,n)$ are the (usual) plane partitions.

By definition, $\mathbb{RPP}(k,n)$ is a pointed rational convex polyhedral cone, with vertex at the origin (corresponding to the empty plane partition). It is the convex hull of $\text{PP}(k,n)$.

Recall that given a linear partition μ , there is a unique plane partition of height 1 having μ as its base. We will use the same notation, μ , for this one-floor plane partition. We consider what we call *chains of floors*: these are sets of non-empty floors which are totally ordered with respect to inclusion. We write them in the form $\mu^\bullet = \{\mu^1 \supsetneq \mu^2 \supsetneq \cdots \supsetneq \mu^h\}$, and call $\ell(\mu^\bullet) = h$ the *length* of the chain μ^\bullet .

Given a chain of floors μ^\bullet of length h and non-negative integers a_1, a_2, \dots, a_h , we have a plane partition $\beta \in \text{PP}(k,n)$ given by

$$\beta = a_1 \mu^1 + a_2 \mu^2 + \cdots + a_h \mu^h.$$

Moreover, any plane partition whose set of floors is contained in μ^\bullet has a unique expression of the above form. We denote by $\text{PP}(\mu^\bullet)$ the set of plane partitions obtained in this way, and by $\mathbb{RPP}(\mu^\bullet)$ the corresponding convex cone.

The following facts follow from straightforward computations:

- (1) A chain of floors μ^\bullet can always be completed to a basis of the lattice $\mathbb{Z}^{k(n-k)}$. In particular the cones $\mathbb{RPP}(\mu^\bullet)$ are non-singular (in the sense of [Ful93, §2.1]), and of dimension $\ell(\mu^\bullet)$.
- (2) Intersections of cones correspond to intersections of chains: given two chains μ^\bullet and ν^\bullet , we have $\mathbb{RPP}(\mu^\bullet) \cap \mathbb{RPP}(\nu^\bullet) = \mathbb{RPP}(\mu^\bullet \cap \nu^\bullet)$. In particular, $\mathbb{RPP}(\nu^\bullet) \subseteq \mathbb{RPP}(\mu^\bullet)$ if and only if $\nu^\bullet \subseteq \mu^\bullet$.
- (3) Given a floor μ , the one-dimensional cone $\mathbb{RPP}(\{\mu\}) = \mathbb{R}_{\geq 0} \cdot \mu$ is an extremal ray of $\mathbb{RPP}(k, n)$.

The collection of cones $\mathbb{RPP}(\mu^\bullet)$, where μ^\bullet ranges among all chains of floors, gives a non-singular fan whose support is $\mathbb{RPP}(k, n)$. The one-dimensional cones in this fan are exactly the extremal rays of $\mathbb{RPP}(k, n)$.

The polytope of normalized Schubert valuations. Given a plane \mathbb{R} -partition $\beta \in \mathbb{RPP}(k, n)$, we denote by $|\beta| = \sum \beta_{i,j}$ the sum of the entries in β , and call it the *volume* of β . Notice that the volume of an element in $\text{PP}(k, n)$ agrees with the log discrepancy of the corresponding valuation. We set

$$\text{SV}(k, n) = \{ \beta \in \mathbb{RPP}(k, n) \mid |\beta| = 1 \},$$

and call it the *polytope of normalized Schubert valuations*. Analogously, if $\mu^\bullet \neq \emptyset$ is a non-empty chain of floors, we set

$$\text{SV}(\mu^\bullet) = \{ \beta \in \mathbb{RPP}(\mu^\bullet) \mid |\beta| = 1 \}.$$

Using the above description of $\mathbb{RPP}(k, n)$ as the support of a non-singular fan, we immediately get a simplicial structure on $\text{SV}(k, n)$. More precisely, $\text{SV}(k, n)$ is a bounded rational convex polytope whose extremal points are of the form $\mu/|\mu|$, where μ ranges among all non-empty linear partitions with at most k parts of size at most $n - k$. For a non-empty chain of floors μ^\bullet , the polytope $\text{SV}(\mu^\bullet)$ is a simplex of dimension $\ell(\mu^\bullet) - 1$, and its faces correspond to sub-chains $\nu^\bullet \subseteq \mu^\bullet$. This collection of simplices endows $\text{SV}(k, n)$ with the structure of a simplicial complex. See Fig. 0C for an example.

We let $\text{Bru}^*(k, n)$ denote the poset of partitions with at most k parts of size $n - k$ endowed with the Bruhat order (containment of partitions), and consider $\text{Bru}(k, n) = \text{Bru}^*(k, n) \setminus \{\emptyset\}$. Then the simplicial complex $\text{SV}(k, n)$ coincides with the nerve (in the sense of category theory) of the poset $\text{Bru}(k, n)$. But notice that $\text{SV}(k, n)$ is not just an abstract simplicial complex, it has a natural geometric realization embedded in $\mathbb{R}^{k(n-k)}$.

The Arnold multiplicity. Fix a Schubert variety Ω_λ , and let $(b_1^{a_1}), \dots, (b_r^{a_r})$ be the Schubert conditions of λ (as defined before Proposition 2.3). Notice that $(a_1, b_1), \dots, (a_r, b_r)$ are the South-East corners of the diagram of λ . It follows from Proposition 3.2 and Definition 3.6 that

$$\text{ord}_\beta(\Omega_\lambda) = \min_{s=1 \dots r} \{ \beta_{a_s, b_s} + \beta_{a_s+1, b_s+1} + \beta_{a_s+2, b_s+2} + \dots \} \quad (10a)$$

for any plane partition β . In the above formula, for each s we are summing the entries of β corresponding to the half-diagonal emanating from the corner (a_s, b_s) . See Fig. 0D for some examples.

We use Eq. (10a) to define $\text{ord}_\beta(\Omega_\lambda)$ when β is a plane \mathbb{R} -partition. We obtain a function on $\mathbb{RPP}(k, n)$, which we denote $\text{ord}(\lambda)$:

$$\text{ord}(\lambda): \mathbb{RPP}(k, n) \rightarrow \mathbb{R}, \quad \beta \mapsto \text{ord}(\lambda)(\beta) = \text{ord}_\beta(\Omega_\lambda).$$

By construction $\text{ord}(\lambda)$ is a concave piecewise-linear function on $\mathbb{RPP}(k, n)$. We denote by H_λ the biggest linear subspace of $\mathbb{R}^{k(n-k)}$ where $\text{ord}(\lambda)$ is linear. This is the biggest linear subspace contained in the corner locus of $\text{ord}(\lambda)$, and its equations are

$$\beta_{a_s, b_s} + \beta_{a_s+1, b_s+1} + \beta_{a_s+2, b_s+2} + \cdots = \beta_{a_{s'}, b_{s'}} + \beta_{a_{s'}+1, b_{s'}+1} + \beta_{a_{s'}+2, b_{s'}+2} + \cdots$$

where s and s' range in $\{1, \dots, r\}$. Observe that when λ is rectangular, $\text{ord}(\lambda)$ is linear, and therefore $H_\lambda = \mathbb{R}^{k(n-k)}$. See Fig. 0D for some examples.

Theorem 10.1. *Let Ω_λ be a Schubert variety in $G(k, n)$. Then the Arnold multiplicity of the pair $(G(k, n), \Omega_\lambda)$ is the maximum of $\text{ord}(\lambda)$ in $\text{SV}(k, n) \cap H_\lambda$.*

Notice that $\text{ord}(\lambda)$ is a linear function on the convex polytope $\text{SV}(k, n) \cap H_\lambda$, and therefore the Arnold multiplicity is achieved at one of the extremal points of the polytope.

Proof. Notice that the contact loci $\text{Cont}^{\geq p}(\Omega_\lambda)$ are unions of contact strata, and in particular their irreducible components are closed contact strata. Therefore, using Eq. (1c) we see that the Arnold multiplicity is given by

$$\text{Arnold-mult}(G(k, n), \Omega_\lambda) = \max_{\beta} \left\{ \frac{\text{ord}_\beta(\Omega_\lambda)}{\text{codim}(\mathcal{C}_\beta, J_\infty G(k, n))} \right\} = \max_{\beta} \left\{ \frac{\text{ord}(\lambda)(\beta)}{|\beta|} \right\},$$

where β ranges among all plane partitions in $\text{PP}(k, n)$. If $\beta \notin H_\lambda$, it is possible to decrease the number of boxes in β without changing $\text{ord}(\lambda)(\beta)$, and we see that the maximum is achieved in $\text{PP}(k, n) \cap H_\lambda$. Observe that $\text{ord}(\lambda)$ is homogeneous:

$$\frac{\text{ord}(\lambda)(\beta)}{|\beta|} = \text{ord}(\lambda) \left(\frac{\beta}{|\beta|} \right).$$

Therefore the Arnold multiplicity is the maximum of $\text{ord}(\lambda)$ on $\text{SV}(k, n) \cap H_\lambda \cap \mathbb{Q}^{k(n-k)}$. The theorem now follows from the fact that $\text{SV}(k, n) \cap H_\lambda$ is a rational polytope. \square

The rectangular case. Theorem 10.1 can be improved slightly when λ is rectangular. That is the content of Theorem 0.4.

Proof of Theorem 0.4. Let $\lambda = (b^a)$ be a rectangular partition. Then $H_\lambda = \mathbb{R}^{k(n-k)}$, and Theorem 10.1 says that the Arnold multiplicity (and therefore the log canonical threshold) is achieved at one of the extremal points of $\text{SV}(k, n)$. These extremal points are of the form $\mu/|\mu|$, where μ is a floor (determined by a linear partition). The theorem follows if we show that we can restrict μ to be in the set $\lambda^0, \dots, \lambda^r$, where $\lambda^s = ((b+s)^{a+s})$ and $r = \min\{k-a, n-k-b\}$.

Let μ be arbitrary, and let s be the biggest index such that $\lambda^s \subset \mu$. Then $\text{ord}(\lambda)(\mu) = \text{ord}(\lambda)(\lambda^s)$ and $|\mu| \geq |\lambda^s|$, and the theorem follows. \square

As we saw in Proposition 2.4, the ideal of a Schubert variety of rectangular shape is essentially equivalent to the ideal of a generic determinantal variety. The log canonical thresholds of determinantal varieties were first computed in [Joh03]. For an approach to the singularities of generic determinantal varieties using arcs, see [Doc13].

$$\begin{aligned}
 &\text{Maximize: } \beta_{1,4} + \beta_{2,5} \\
 &\text{Subject to: } \beta_{1,4} + \beta_{2,5} = \beta_{2,2} + \beta_{3,3} \\
 &\quad \beta_{1,4} + \beta_{2,5} = \beta_{3,1} \\
 &\quad \sum_{i=1..3} \sum_{j=1..5} \beta_{i,j} = 1 \\
 &\quad \beta_{i,j} \geq \beta_{i+1,j} \quad \forall i \in \{1,2\} \quad \forall j \in \{1,2,3,4,5\} \\
 &\quad \beta_{i,j} \geq \beta_{i,j+1} \quad \forall i \in \{1,2,3\} \quad \forall j \in \{1,2,3,4\} \\
 &\quad \beta_{1,1} \geq 1 \quad \beta_{1,2} \geq 1 \quad \beta_{1,3} \geq 1 \quad \beta_{1,4} \geq 1 \quad \beta_{1,5} \geq 0 \\
 &\quad \beta_{2,1} \geq 1 \quad \beta_{2,2} \geq 1 \quad \beta_{2,3} \geq 0 \quad \beta_{2,4} \geq 0 \quad \beta_{2,5} \geq 0 \\
 &\quad \beta_{3,1} \geq 1 \quad \beta_{3,2} \geq 0 \quad \beta_{3,3} \geq 0 \quad \beta_{3,4} \geq 0 \quad \beta_{3,5} \geq 0
 \end{aligned}$$

 FIGURE 10A. The linear program for $\lambda = (421)$ in $G(3,8)$.

Linear programming. Theorem 10.1 does not give a closed formula for the Arnold multiplicity. To get an actual value, one needs to solve a linear programming problem: maximize the linear function $\text{ord}(\lambda)$ on the polytope $\text{SV}(k, n) \cap H_\lambda$. In the present case, we believe this is a task better left to a computer. The equations defining $\text{SV}(k, n) \cap H_\lambda$ and $\text{ord}(\lambda)$ are easy to describe to a computer, and the complexity of the resulting linear program, which is high when approached manually, is perfectly manageable by modern machines. See Fig. 10A for an example of the input that would be fed to a linear programming solver. Notice that in Fig. 10A we have added the constraints $\beta_{i,j} \geq 1$ for $(i, j) \in \lambda$. This is done so the computer does not need to spend time searching for an initial extremal point of the polytope, and can focus on just maximizing the objective function.

In small cases, it is possible to run a linear programming solver by hand. We sketch the idea of the standard algorithm (the simplex method), which is straightforward. We start with β^1 , the one-floor plane partition with base λ . Notice that $\beta^1/|\beta^1|$ is an extremal point of $\text{SV}(k, n) \cap H_\lambda$. Assuming we have constructed β^s , we try to find β^{s+1} verifying

$$\frac{|\beta^{s+1}|}{s+1} < \frac{|\beta^s|}{s}. \quad (10b)$$

The possible candidates β^{s+1} are obtained from β^s by first adding one box to each of the half-diagonals determined by λ (see Fig. 0D), and then completing with more boxes away from the half-diagonals in order to obtain a plane partition. If none of the candidates verifies Eq. (10b), we stop the algorithm and the log canonical threshold is $|\beta^s|/s$. Otherwise we choose β^{s+1} such that $|\beta^{s+1}|/(s+1)$ is minimal among all the candidates. In this process, $\beta^s/|\beta^s|$ iterates among extremal points of the polytope $\text{SV}(k, n) \cap H_\lambda$, and therefore the algorithm finishes after a finite number of steps. An example of the execution of this algorithm appears in Fig. 10B.

From the above description of the algorithm, we immediately get log canonical thresholds of many partitions “with small number of boxes”. To make this precise, consider a partition λ with at most $k-1$ parts of size at most $n-k-1$. Then let $r(\lambda)$ be the *rim size* of λ : the number of boxes in the rectangle $k \times (n-k)$ which touch λ (possibly just in a vertex) but are not contained in λ . Then, if $|\lambda| \leq r(\lambda)$, one can see that the algorithm stops at β^1 . Therefore:

$$\text{lct}(G(k, n), \Omega_\lambda) = |\lambda| \quad \Leftrightarrow \quad |\lambda| \leq r(\lambda).$$

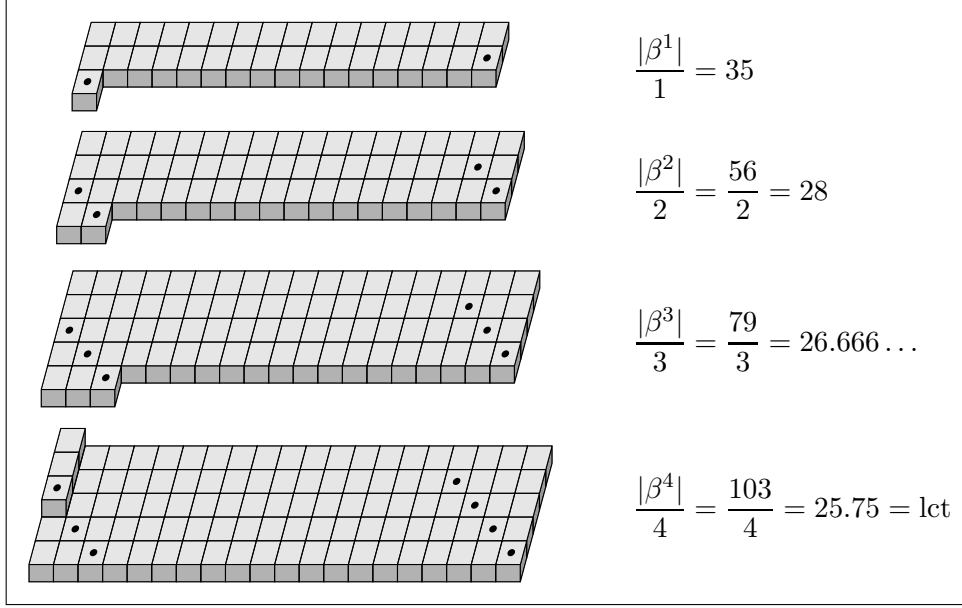


FIGURE 10B. Plane partitions visited by the linear programming algorithm for $\lambda = (17, 17, 1)$ in any $G(k, n)$ with $k \geq 5$ and $n - k \geq 20$.

Notice that $|\lambda| = \text{codim}(\Omega_\lambda, G(k, n))$ is the maximal possible value for the log canonical threshold. In a sense, the Schubert varieties Ω_λ for which $|\lambda| \leq r(\lambda)$ are the least singular.

Log resolutions. An alternative, more direct, approach for the computation of log canonical thresholds would be to use the definition with log resolutions. Unfortunately, we do not know a usable description of log resolutions for all Schubert varieties.

A natural candidate would be the one provided in [Bou93]. The construction resembles the one for generic determinantal varieties. We start with the Grassmannian $X_0 = G(k, n)$ and we let X_1 be the blowing-up of X_0 along the Schubert point (the Borel-fixed point). Then X_2 is the blowing up of X_1 along the strict transform of the one-dimensional Schubert variety. In X_2 , the strict transforms of the two-dimensional Schubert varieties are smooth and disjoint, so we can blow them up (in any order) and obtain X_3 . Recursively, X_{s+1} is obtained from X_s by blowing up the strict transforms of the s -dimensional Schubert varieties (which, as is shown in [Bou93], are smooth and disjoint in X_s). At the end we obtain $X = X_{k(n-k)-1}$. The variety X has $\binom{n}{k} - 1$ exceptional divisors, each one corresponding to a Schubert variety in $G(k, n)$ (except $G(k, n)$ itself). We have not worked this out in complete detail, but there is strong evidence suggesting that the polytope $\text{SV}(k, n)$ is the dual complex of the resolution $X \rightarrow G(k, n)$.

In [Bou93] it is not studied whether X is a log resolution for all Schubert varieties: we do not know if the exceptional locus is a simple normal crossings divisor, and, more importantly, if the scheme-theoretic inverse image of Ω_λ is a divisor. In fact, it seems that X is not a log resolution, at least for some of Schubert varieties. If it were, the log canonical threshold would be computed by one of the valuations corresponding to the exceptional divisors in X . In terms of the arc space, these valuations correspond to one-floor plane partitions. But there are Schubert varieties for which the log canonical

threshold is computed by a plane partition with more than one floor. We already saw one example in Fig. 10B: $\Omega_{(17,17,1)}$ in $G(5, 25)$. An smaller example is $\Omega_{(54441)}$ in $G(5, 10)$.

It would be interesting to construct a simultaneous log resolution for all Schubert varieties in $G(k, n)$. The dual complex of such resolution would be a simplicial subdivision of $SV(k, n)$, in such a way that all the slices $SV(k, n) \cap H_\lambda$ would be simplicial sub-complexes.

11. The Nash problem for Schubert varieties

In this section we study the Nash problem for Schubert varieties in the Grassmannian. The results here are largely independent from the rest of the paper, as it turns out that the structure of the arc space plays a small role towards the solution of the Nash problem.

The main tool that we use are (a particular instance of) the Zelevinsky resolutions. We show that their exceptional components are in bijection with the irreducible components of the exceptional locus.

Partition λ	Outside corners	Singular components $\lambda^1, \dots, \lambda^r$
	(0,9) (2,8) (5,5) (6,3) (7,0)	
	(3,9) (4,5) (7,4)	
	(3,9) (7,4)	None

FIGURE 11A. Some examples of outside corners of partitions in $G(7, 16)$.

The singular locus. Before studying resolutions, we need to recall what is the singular locus of a Schubert variety, and fix some notations. We let Ω_λ be a Schubert variety in $G(k, n)$. A *proper outside corner* of λ is a pair (a, b) , such that the partition (b^a) is a Schubert condition of λ (as it was discussed in Section 2). In other words, we say that (a, b) is a proper outside corner of λ if the rectangle of size $a \times b$ is a maximal sub-rectangle of the diagram of λ . If no proper outside corner of λ is of the form $(a, n - k)$, we say that $(0, n - k)$ is a *virtual outside corner* of λ . Analogously, $(k, 0)$ is a *virtual outside corner* of λ if no proper outside corner is of the form (k, b) . An *outside corner* of λ is either a proper outside corner or a virtual outside corner of λ .

With λ fixed, we denote the outside corners of λ by:

$$(a_0, b_0), \quad (a_1, b_1), \quad \dots, \quad (a_r, b_r), \quad (a_{r+1}, b_{r+1}).$$

Here we assume that the corners are ordered from North-East to South-West, that is:

$$a_0 < a_1 < \dots < a_r < a_{r+1}, \quad \text{and} \quad b_0 > b_1 > \dots > b_r > b_{r+1}.$$

These outside corners determine completely the partition λ , and therefore the Schubert variety Ω_λ . In fact, as we saw in Proposition 2.3, a point $V \in G(k, n)$ belongs to Ω_λ

precisely when

$$\dim V \cap F_{n-k+a_s-b_s} \geq a_s$$

for all $0 \leq s \leq r+1$. Here F_\bullet is the complete flag fixed by the Borel subgroup, as usual.

For $1 \leq s \leq r$, we construct a partition λ^s from λ by adding a rim of boxes around the corner (a_s, b_s) . More precisely, the diagram of λ^s is the union of the diagram of λ and the diagram of the rectangular partition with proper outside corner (a_s+1, b_s+1) . Notice that when $r = -1$ (which only happens when $\lambda = ((n-k)^k)$) and when $r = 0$ we do not construct any partition λ^s . For examples, see Fig. 11A.

Notice that the Borel subgroup acts on Ω_λ , and therefore the singular locus $\text{Sing}(\Omega_\lambda)$ must be Borel-invariant, i.e., it is a union of Schubert varieties. The next theorem identifies the irreducible components of $\text{Sing}(\Omega_\lambda)$ as the Schubert varieties given by the partitions $\lambda^1, \dots, \lambda^r$ defined above. For a proof we refer the reader to [LW90, Thm. 5.3], or to [BV88, Sec. 6.B].

Theorem 11.1. *With the notations introduced above, the singular locus of the Schubert variety Ω_λ is given by:*

$$\text{Sing}(\Omega_\lambda) = \Omega_{\lambda^1} \cup \dots \cup \Omega_{\lambda^r}.$$

In particular, λ is smooth if and only if $r \leq 0$, and otherwise $\text{Sing}(\Omega_\lambda)$ has r irreducible components. In general, a point $V \in G(k, n)$ belongs to the smooth locus of Ω_λ if and only if

$$\dim V \cap F_{n-k+a_s-b_s} = a_s$$

for each outside corner (a_s, b_s) of λ .

Zelevinsky resolutions. In [Zel83], Zelevinsky gives a construction of several resolutions of singularities of Ω_λ , and shows that some of them are small resolutions in the sense of intersection homology. For our purposes, the smallness of the resolution has no relevance, so we will focus in a particular case, which is easy to describe.

We consider the manifold $\text{Fl}(a_1, a_2, \dots, a_r, k; n)$ of partial flags in \mathbb{C}^n of the form

$$U_1 \subset U_2 \subset \dots \subset U_r \subset V \subset \mathbb{C}^n,$$

where

$$\dim U_s = a_s, \quad \text{and} \quad \dim V = k.$$

We let $Y \subset \text{Fl}(a_1, a_2, \dots, a_r, k; n)$ be the subvariety corresponding to those flags that verify

$$U_s \subseteq F_{n-k+a_s-b_s}$$

for all $1 \leq s \leq r$. It is straightforward to check that Y is a tower of Grassmann bundles, and in particular it is smooth and irreducible.

There is a natural projection $f: Y \rightarrow G(k, n)$ obtained by sending a flag $U_1 \subset \dots \subset U_r \subset V$ to just V . For any flag in Y , we have that $U_s \subseteq V \cap F_{n-k+a_s-b_s}$, and therefore $\dim V \cap F_{n-k+a_s-b_s} \geq a_s$. This shows that the image of f is exactly $f(Y) = \Omega_\lambda$. Moreover, for V in the smooth locus of Ω_λ , Theorem 11.1 shows that there is exactly one flag in Y mapping to V , the one for which $U_s = V \cap F_{n-k+a_s-b_s}$. In other words, the morphism $f: Y \rightarrow \Omega_\lambda$ is a resolution of singularities, and f is an isomorphism over the smooth locus of Ω_λ .

We are now ready to prove the main result of this section.

Proposition 11.2. *Let Ω_λ be a Schubert variety in $G(k, n)$, and let $f: Y \rightarrow \Omega_\lambda$ be the resolution of singularities described above. Then the exceptional components of f are in bijection with the irreducible components of $\text{Sing}(\Omega_\lambda)$.*

Proof. Fix $1 \leq s \leq r$. We consider the subvariety

$$Z_s \subset \text{Fl}(a_1, \dots, a_s, a_s + 1, a_{s+1}, \dots, a_r, k; n)$$

given by those flags

$$U_1 \subset \dots \subset U_s \subset U_s^* \subseteq U_{s+1} \subset \dots \subset U_r \subset V \subset \mathbb{C}^n$$

that verify

$$\dim U_1 = a_1, \quad \dots, \quad \dim U_r = a_r, \quad \dim V = k, \quad \dim U_s^* = a_s + 1,$$

and

$$U_1 \subseteq F_{n-k+a_1-b_1}, \quad \dots, \quad U_r \subseteq F_{n-k+a_r-b_r}, \quad U_s^* \subseteq F_{n-k+a_s-b_s}.$$

Similar considerations as the ones discussed above for Y show that Z_s is smooth and, more importantly for us, irreducible. There is a natural morphism $g_s: Z_s \rightarrow Y$, and the image of the composition $f \circ g_s$ is Ω_{λ^s} , one of the irreducible components of the singular locus of Ω_{λ} . We denote by $E_s \subset Y$ the image of the morphism g_s . Notice that each of the E_s is irreducible. We have shown that $f(E_1 \cup \dots \cup E_r) = \text{Sing}(\Omega_{\lambda})$.

Consider now a flag $U_1 \subset \dots \subset U_r \subset V$ in Y , and assume that $V \in \text{Sing}(\Omega_{\lambda})$. Let s be the largest index such that $\dim V \cap F_{n-k+a_s-b_s} > a_s$. We know that such s exists, because $V \in \text{Sing}(\Omega_{\lambda})$. Let U_s^* be any vector space of dimension $a_s + 1$ containing U_s and contained in $V \cap F_{n-k+a_s-b_s}$. From the construction of s , we must have that either $s = r$ or otherwise

$$U_{s+1} = V \cap F_{n-k+a_{s+1}-b_{s+1}} \supseteq U_s^*.$$

Therefore U_s^* can be used to define a flag in Z_s , and we see that the original flag $U_1 \subset \dots \subset U_r \subset V$ belongs to E_s . We have shown that $f^{-1}(\text{Sing}(\Omega_{\lambda})) = E_1 \cup \dots \cup E_r$, as required. \square

Corollary 11.3. *Let Ω_{λ} be a Schubert variety in $G(k, n)$. Then the Nash map for Ω_{λ} is bijective.*

Proof. Recall from Section 1 that the Nash families are the irreducible components of $\text{Cont}^{\geq 1}(\text{Sing}(\Omega_{\lambda}))$, and that the Nash map is an injection that associates to each Nash family an essential valuation. Since $\text{Sing}(\Omega_{\lambda})$ has r irreducible components, we see that there are at least r Nash families. Proposition 11.2 shows that there are at most r essential valuations. The corollary follows from the fact that the Nash map is injective. \square

Nash valuations. For a Schubert variety Ω_{λ} in $G(k, n)$, the Nash valuations (which, as we saw above, agree with the essential valuations) can be described using contact strata. Recall that the arc space of Ω_{λ} is the union of contact strata:

$$J_{\infty}\Omega_{\lambda} = \bigcup_{\beta} \mathcal{C}_{\beta},$$

where β ranges among all plane partitions that have infinite height on λ :

$$\beta_{i,j} = \infty \quad \forall (i, j) \in \lambda.$$

As above, consider the partitions $\lambda^1, \dots, \lambda^r$ giving the irreducible components of the singular locus:

$$\text{Sing}(\Omega_{\lambda}) = \Omega_{\lambda^1} \cup \dots \cup \Omega_{\lambda^r}.$$

For each $1 \leq s \leq r$, we let β^s be the plane partition with bottom floor equal to λ^s and an infinite number of floors equal to λ . In other words, β^s is given by

$$\beta_{i,j}^s = \begin{cases} \infty & \text{if } (i,j) \in \lambda, \\ 1 & \text{if } (i,j) \in \lambda^s \text{ and } (i,j) \notin \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows from the above discussion on the Nash problem that the Nash/essential valuations for Ω_λ are precisely

$$\text{ord}_{\beta^1}, \text{ord}_{\beta^2}, \dots, \text{ord}_{\beta^r}.$$

Notice that these are only semi-valuations on $G(k,n)$ (they have infinite terms), but they are valuations on Ω_λ .

We would like to remark that it is also possible to show directly that

$$\text{Cont}^{\geq 1}(\text{Sing}(\Omega_\lambda)) = \overline{\mathcal{C}}_{\beta^1} \cup \dots \cup \overline{\mathcal{C}}_{\beta^r},$$

without using resolutions. This is in fact an easy consequence of Theorem 8.5.

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Roi Docampo

Instituto de Ciencias Matemáticas (ICMAT)
c/ Nicolás Cabrera, 13–15
Campus de Cantoblanco, UAM
28049 Madrid, Spain
roi.docampo@icmat.es

Antonio Nigro

Instituto de Matemática e Estatística
Universidade Federal Fluminense
Rua Mário Santos Braga, s/n
24020–140 Niterói, RJ, Brasil
nigro@impa.br